

# Constrained Triangulations, Volumes of Polytopes, and Unit Equations

Michael Kerber, Robert Tichy, Mario Weitzer

September 20, 2016

## Abstract

Given a polytope  $\mathcal{P}$  in  $\mathbb{R}^d$  and a subset  $U$  of its vertices, is there a triangulation of  $\mathcal{P}$  using  $d$ -simplices that all contain  $U$ ? We answer this question by proving an equivalent and easy-to-check combinatorial criterion for the facets of  $\mathcal{P}$ . Our proof relates triangulations of  $\mathcal{P}$  to triangulations of its “shadow”, a projection to a lower-dimensional space determined by  $U$ . In particular, we obtain a formula relating the volume of  $\mathcal{P}$  with the volume of its shadow. This leads to an exact formula for the volume of a polytope arising in the theory of unit equations.

## 1 Introduction

**Problem statement and results.** Let  $\mathcal{P}$  be a convex polytope in  $\mathbb{R}^d$ , that is, the convex hull of a finite point set  $V$ , and let  $U$  be a subset of  $V$ . We ask for a triangulation of (the interior of)  $\mathcal{P}$  with the property that every  $d$ -simplex in the triangulation contains all points of  $U$  as vertices, calling it a  *$U$ -spinal triangulation*. A simple example is the *star triangulation* of  $\mathcal{P}$  (Figure 1), where all  $d$ -simplices contain a common vertex  $\mathbf{p}$ , and  $U$  is the singleton set consisting of that point. Another example is the  $d$ -hypercube with  $U$  being a pair of opposite points (Figure 2). Indeed, the hypercube can be triangulated in a way that all  $d$ -simplices contain the space diagonal spanned by  $U$  [13, 8].

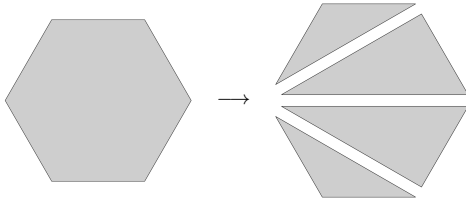


Figure 1: A star triangulation of a hexagon.

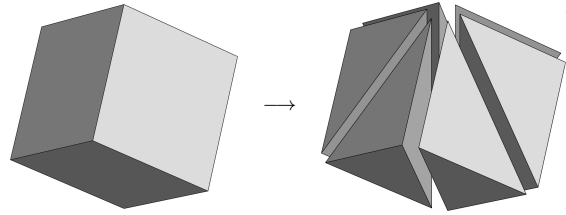


Figure 2: A  $U$ -spinal triangulation of a cube, where  $U$  consists of the two vertices on a space diagonal.

We are interested in what combinations of  $\mathcal{P}$  and  $U$  admit spinal triangulations. One of our main results is a simple combinatorial answer for this question: Denoting by  $n$  the cardinality of  $U$ , we prove that a  $U$ -spinal triangulation of  $\mathcal{P}$  exists if and only if each facet of  $\mathcal{P}$  contains at least  $n - 1$  vertices of  $U$ . Moreover, we provide a complete characterization of such spinal triangulations: let  $\Phi$  denote the orthogonal projection of  $\mathbb{R}^d$  to the orthogonal complement of the lower-dimensional flat spanned by  $U$ .  $\Phi$  maps  $U$  to  $\mathbf{0}$  by construction, and  $\mathcal{P}$  is mapped to a *shadow*  $\hat{\mathcal{P}} := \Phi(\mathcal{P})$ . We obtain a  $U$ -spinal triangulation of  $\mathcal{P}$  by first star-triangulating  $\hat{\mathcal{P}}$  with respect to  $\mathbf{0}$  and then lifting each maximal simplex to  $\mathbb{R}^d$  by taking the preimage of its vertices under  $\Phi$  (Figure 3). Vice versa, every spinal triangulation can be obtained in this way.

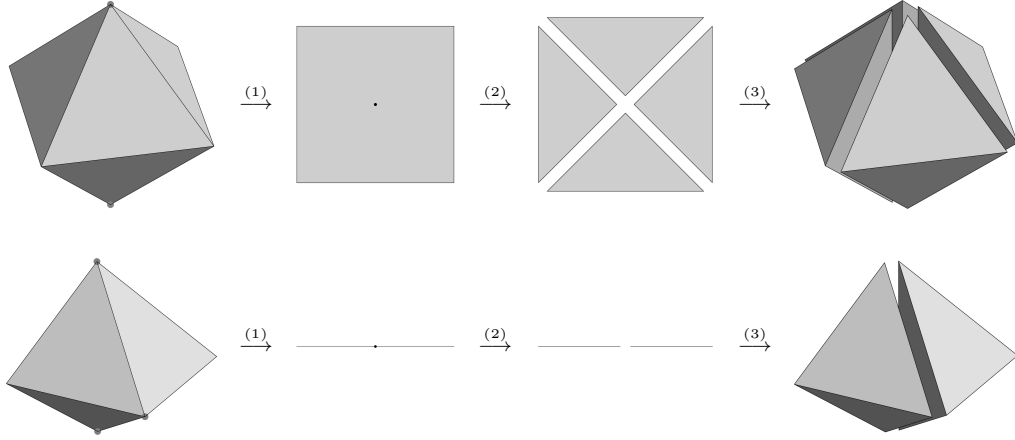


Figure 3: Two examples of the lifting process: (1) Project polytope  $\mathcal{P}$  to the orthogonal complement of the flat spanned by  $U$  (vertices marked by prominent dots) to obtain shadow  $\hat{\mathcal{P}}$ . (2) Star-triangulate  $\hat{\mathcal{P}}$  with respect to the origin. (3) Lift star triangulation of  $\hat{\mathcal{P}}$  to obtain  $U$ -spinal triangulation of  $\mathcal{P}$ . Note: every facet of  $\mathcal{P}$  contains exactly  $|U| - 1$  points of  $U$  in both examples.

An interesting consequence of our characterization is that a spine allows us to relate the volumes of a convex polytope  $\mathcal{P}$  and its shadow  $\hat{\mathcal{P}}$  (with respect to that spine) by a precise equation. An application of this observation leads to our other main result: an exact volume formula of an important polytope arising in number theory which we call the *Everest polytope*. We show that this polytope is the shadow of a higher-dimensional *simplotope*, the product of simplices, whose volume is easy to determine.

**Number theoretic background.** In the following we briefly discuss the number-theoretic background of the Everest polytope. G. R. Everest [9, 10] studied various counting problems related to Diophantine equations. In particular, he proved asymptotic results for the number of values taken by a linear form whose variables are restricted to lie inside a given finitely generated subgroup of a number field. This includes norm form- and discriminant form equations, normal integral bases and related objects. Everest's work contains important contributions to the quantitative theory of  $S$ -unit equations and makes use of Baker's theory of linear forms in logarithms and Schmidt's subspace theorem from Diophantine approximation; see for instance [25, 11, 1]. Later, other authors [12] applied the methods of Everest to solve combinatorial problems in algebraic number fields. The corresponding counting results involve various important arithmetic constants, one of them being the volume of a certain convex polytope.

In order to introduce Everest's constant, we use basic facts from algebraic number theory. Let  $K$  be a number field,  $N = N_{K/\mathbb{Q}}$  the field norm and  $S$  a finite set of places of  $K$  including the archimedean ones. We denote by  $O_{K,S} = \{\alpha \in K : |\alpha|_v \leq 1 \text{ for all } v \notin S\}$  the ring of  $S$ -integers and its unit group by  $U_{K,S}$ ; the group of  $S$ -units. Let  $c_0, \dots, c_n$  denote given non-zero algebraic numbers. During the last decades, a lot of work is devoted to the study of the values taken by the expression  $c_0x_0 + \dots + c_nx_n$ , where the  $x_n$  are allowed to run through  $U_{K,S}$ , see for instance [23, 17]. A specific instance of this kind of general  $S$ -unit equations is the following combinatorial problem. As usual, two  $S$ -integers  $\alpha$  and  $\beta$  are said to be associated (for short  $\alpha \sim \beta$ ) if there exists an  $S$ -unit  $\epsilon$  such that  $\alpha = \beta\epsilon$ . It is well-known that the group of  $S$ -units  $U_{K,S}$  is a free abelian group with  $s = |S| - 1$  generators,  $\omega_K$  and  $\text{Reg}_{K,S}$  denote as usual the number of roots of unity and the  $S$ -regulator of  $K$ , respectively (for the basic concepts of algebraic number theory see [22]). Then for given  $n \in \mathbb{N}$ ,  $q > 0$  the counting function  $u(n, q)$  is defined as the number of equivalence classes  $[\alpha]_{\sim}$  such that

$$N(\alpha) := \prod_{v \in S} |\alpha|_v \leq q, \quad \alpha = \sum_{i=1}^n \varepsilon_i,$$

where  $\varepsilon_i \in U_{K,S}$  and no subsum of  $\varepsilon_1 + \dots + \varepsilon_n$  vanishes. From the work of Everest [9, 10], the following asymptotic formula can be derived:

$$u(n, q) = \frac{c(n-1, s)}{n!} \left( \frac{\omega_K(\log q)^s}{\text{Reg}_{K,S}} \right)^{n-1} + o((\log q)^{(n-1)s-1+\varepsilon})$$

for arbitrary  $\varepsilon > 0$ . Here  $c(n-1, s)$  is a positive constant, and its exact value is only known in very special cases; see [2] for more details. In general,  $c(n, s)$  is given as the volume of a convex polytope in  $\mathbb{R}^{ns}$  that we define and study in Section 3. Our results show that

$$c(n, s) = \frac{1}{(s!)^{n+1}} \frac{((n+1)s)!}{(ns)!},$$

which can also be written in terms of a multinomial coefficient as  $\binom{ns}{s, \dots, s} \frac{1}{(ns)!}$ .

**Geometric background.** Our results fall into the category of *constrained triangulations of convex polytopes*. Triangulations of polytopes are a classic topic in discrete geometry; an infamous question is the quest for triangulating a  $d$ -hypercube with a minimal number of simplices [4]. Precise answers are only known up to dimension 7 using computer-assisted proofs [19]. A contemporary overview on results relating to triangulations of polytopes and more general point configurations is provided by de Loera, Rambau, and Santos [7]. It includes a discussion on the triangulation of simplotopes, a geometric object whose study goes back to Hadwiger [18], and has been studied, for instance, in the context of combinatorics [14], game theory [28] and algebraic geometry [16, Ch. 7]. Simplotopes admit a standard triangulation, the so-called *staircase triangulation*, which can easily be described in combinatorial terms. A by-product of our results is that simplotopes can also be triangulated by a family of spinal triangulations; we refer to Section 7 for details.

Constraining triangulations has mostly been considered for low-dimensional problems under an algorithmic angle. For instance, a *constrained Delaunay triangulation* is a triangulation which contains a fixed set of pre-determined simplices; apart from these constraints, it tries to be “as Delaunay as possible”; see Shewchuk’s work [27] for details. While our work is related in spirit, there appears to be no direct connection to this framework, because our constraint does not only ensure the presence of certain simplices in the triangulation, but rather constrains all  $d$ -simplices at once.

Computing volumes of high-dimensional convex polytopes is another notoriously hard problem, from a computational perspective [21, Sec. 13] as well as for special cases. A famous example is the *Birkhoff polytope*, whose exact volume is known exactly only up to dimension 10 [24, 6]. Our contribution provides a novel technique to compute volumes of polytopes through lifting into higher dimensions. We point out that lifting increases the dimension, so that the lift of a polytope is not the image of a linear transformation. Therefore, the well-known formula  $\text{vol}(A\mathcal{P}) = \sqrt{\det(A^T A)} \text{vol}(\mathcal{P})$  with  $A \in \mathbb{R}^{e \times d}$  and  $e \geq d$  does not apply to our case. Another interpretation of our technique is to relate the volume of a polytope and the volume of its (spinal) projection. For a fixed normal vector  $\mathbf{u}$ , the volume  $\text{vol}_{\mathbf{u}}$  of the projection of a polytope to the hyperplane normal to  $\mathbf{u}$  is given by

$$\text{vol}_{\mathbf{u}} = \frac{1}{2} \sum_{i=1}^m |\mathbf{u} \tilde{\mathbf{u}}_i|,$$

where  $\tilde{\mathbf{u}}_1, \dots, \tilde{\mathbf{u}}_m$  are the outward normal vectors of the facets of the polytope scaled by their volume [3, Thm 1.1]. The cited paper also discusses the algorithmic problem of finding the projection vector that yields the maximal and minimal projected volume. More results relating the volume of a polytope with its projections (in codimension 1) are studied under the name of *geometric tomography* [15]; see also [26]. We do not see a simple way to derive the volume of a spinal projection using these approaches, partially because our result relates the volume of polytopes of larger codimension.

We emphasize that our result holds for arbitrary (bounded) convex polytopes, avoiding simplifying assumptions such as being simple or simplicial. Nevertheless, we require simplicial position of the polytope for some arguments in our proofs, and we face the challenge of arguing that we can pass to a simplicial polytope without loss of generality. This will require a careful study of how a polytope changes when its vertices are perturbed by a small amount. For that purpose, we make use of the technique of *push- and pull-perturbations*, which are also discussed in detail by de Loera et al. [7].

**Organization.** We start by introducing the basic concepts from convex geometry in Section 2. Then, we proceed with the prove of our number-theoretic formula: we calculate the vertices of the Everest polytope in Section 3 and define a map from a simplotope to the Everest polytope in Section 4. Then, we establish the volume formula for the Everest polytope in Section 5 by proving our structural result on spinal triangulations under additional assumptions which are satisfied for the Everest polytope. We remove these additional assumptions in Section 6, proving our characterization of spinal triangulations in generality. We conclude with some additional remarks in Section 7.

## 2 Geometric concepts

Let  $M$  be an arbitrary subset of  $\mathbb{R}^d$  with some integer  $d \geq 1$ . The *dimension* of  $M$  is the dimension of the smallest affine subspace of  $\mathbb{R}^d$  containing  $M$ . We say that  $M$  is *full-dimensional* if its dimension is equal to  $d$ . Throughout the entire paper,  $V$  will always stand for a finite point set in  $\mathbb{R}^d$  that is full-dimensional and in *convex position*, that is  $x \notin \text{conv}(V \setminus \{x\})$  for every  $x \in V$ , where  $\text{conv}(\cdot)$  denotes the *convex hull* in  $\mathbb{R}^d$ .

**Polytopes and simplicial complexes.** We use the following standard definitions (compare, for instance, Ziegler [29]): A *polytope*  $\mathcal{P}$  is the convex hull of a finite point set in  $\mathbb{R}^d$  in which case we say that the point set *spans*  $\mathcal{P}$ . A hyperplane  $\mathbb{H} \subseteq \mathbb{R}^d$  is called *supporting* (for  $\mathcal{P}$ ) if  $\mathcal{P}$  is contained in one of the closed half-spaces induced by  $\mathbb{H}$ . A *face* of  $\mathcal{P}$  is either  $\mathcal{P}$  itself, or the intersection of  $\mathcal{P}$  with a supporting hyperplane. If a face is neither the full polytope nor empty, we call it *proper*. A face of dimension  $\ell$  is also called  $\ell$ -face of  $\mathcal{P}$ , with the convention that the empty set is a  $(-1)$ -face. We call the union of all proper faces of  $\mathcal{P}$  the *boundary* of  $\mathcal{P}$ , and the points of  $\mathcal{P}$  not on the boundary the *interior* of  $\mathcal{P}$ . 0-faces are called the *vertices* of  $\mathcal{P}$ , and we let  $V(\mathcal{P})$  denote the set of vertices. With  $\ell$  being the dimension of  $\mathcal{P}$ , we call  $(\ell - 1)$ -faces *facets*, and  $(\ell - 2)$ -faces *ridges* of  $\mathcal{P}$ . Any face  $\mathcal{F}$  of  $\mathcal{P}$  is itself a polytope whose vertex set is  $V(\mathcal{P}) \cap \mathcal{F}$ .

If  $\mathcal{P}$  is  $d$ -dimensional and  $\mathcal{F}$  a facet of  $\mathcal{P}$ , there is a unique supporting hyperplane for  $\mathcal{P}$  that contains  $\mathcal{F}$ , which we denote as the *support* of  $\mathcal{F}$ . If  $\mathcal{P}$  is of dimension  $\ell < d$ , we can consider it as a polytope in  $\mathbb{R}^\ell$  by passing to the  $\ell$ -dimensional affine subspace spanned by  $\mathcal{P}$ . Within this subspace, again, it makes sense to talk about the support of a facet, which is an affine subspace of dimension  $\ell - 1$ .

It is well-known that every point  $\mathbf{p} \in \mathcal{P}$  (and only those) can be written as a *convex combination* of vertices of  $\mathcal{P}$ , that is  $\mathbf{p} = \sum_{\mathbf{v} \in V(\mathcal{P})} \lambda_{\mathbf{v}} \mathbf{v}$  with real values  $\lambda_{\mathbf{v}} \geq 0$  for all  $\mathbf{v}$  and  $\sum_{\mathbf{v} \in V(\mathcal{P})} \lambda_{\mathbf{v}} = 1$ . By Caratheodory's theorem, there exists a convex combination with at most  $d + 1$  non-zero entries, that is,  $\mathbf{p} = \sum_{i=1}^{d+1} \lambda_i \mathbf{v}_i$  with  $\mathbf{v}_i \in V(\mathcal{P})$ ,  $\lambda_i \geq 0$  and  $\sum \lambda_i = 1$ .

An  $\ell$ -simplex  $\sigma$  with  $\ell \in \{-1, \dots, d\}$  is a polytope of dimension  $\ell$  that has exactly  $\ell + 1$  vertices. Every point in a simplex is determined by a unique convex combination of the vertices. We call a polytope *simplicial* if all its proper faces are simplices. In this case, we call the vertex set of the polytope to be *in simplicial position*. A *simplicial complex*  $\mathcal{C}$  in  $\mathbb{R}^d$  is a set of simplices in  $\mathbb{R}^d$  such that for a simplex  $\sigma$  in  $\mathcal{C}$ , all faces of  $\sigma$  are in  $\mathcal{C}$  as well, and if  $\sigma$  and  $\tau$  are in  $\mathcal{C}$ , the intersection  $\sigma \cap \tau$  is a common face of both (note that the empty set is a face of any polytope). We let  $V(\mathcal{C})$  denote the set of all vertices in  $\mathcal{C}$ . The *underlying space*  $\bigcup \mathcal{C}$  of  $\mathcal{C}$  is the union of its simplices. We call a simplex in  $\mathcal{C}$  *maximal* if it is not a proper face of another simplex in  $\mathcal{C}$ . A complex equals the set of its maximal simplices together with all their faces and is therefore uniquely determined by its maximal simplices. Also, the underlying space of  $\mathcal{C}$  equals the union of its maximal simplices.

In what follows, we let  $\mathcal{P} := \text{conv}(V)$  be the polytope spanned by  $V$  as fixed above. In particular,  $\dim(\mathcal{P}) = d$  because  $V$  is full-dimensional and  $V(\mathcal{P}) = V$  because  $V$  is in convex position.

**Spines.** We call  $U \subseteq V$  with  $|U| = n$  a *spine* of  $V$  if each facet of  $\mathcal{P}$  contains at least  $n - 1$  points of  $U$ . We call  $U$  a *strong spine* of  $V$  if each facet has exactly  $n - 1$  points of  $U$ . Trivially, a one-point subset of  $V$  is a spine, but not a strong spine. Similarly, if  $\mathcal{P}$  is a simplex, any non-empty subset of vertices is a spine, but not a strong spine except for  $U = V$ . For a hypercube, every pair of opposite vertices forms a strong spine, but no other spines with two or more elements exists.

We derive an equivalent geometric characterization of spines next. The  $U$ -span (in  $V$ ) is the set of all  $d$ -simplices  $\sigma$  satisfying  $U \subseteq V(\sigma) \subseteq V$ . Equivalently, it is the set of all  $d$ -simplices with vertices in  $V$  which have  $\text{conv}(U)$  as a common face. Clearly, each simplex  $\sigma$  of the  $U$ -span is contained in  $\mathcal{P}$ , and the same is true for the union of all simplices in the  $U$ -span.

**Lemma 1.** *Let  $U \subseteq V$  with  $|U| = n$ . Then,  $U$  is a spine of  $V$  if and only if the union of all  $U$ -span simplices is equal to  $\mathcal{P}$ , that is, if every point in  $\mathcal{P}$  belongs to at least one simplex in the  $U$ -span.*

*Proof.* We prove both directions of the equivalence separately. For “ $\Rightarrow$ ”, we proceed by induction on  $n$ . The statement is true for  $n = 0$  by Caratheodory’s theorem. Let  $U$  be a set with at least one element,  $\mathbf{u} \in U$  arbitrary, and  $\mathbf{p} \in \mathcal{P} \setminus \{\mathbf{u}\}$ . The ray starting in  $\mathbf{u}$  through  $\mathbf{p}$  leaves the polytope in a point  $\bar{\mathbf{p}}$ , and this point lies on (at least) one facet  $\mathcal{F}$  of  $\mathcal{P}$  that does not contain  $\mathbf{u}$ . By assumption,  $\mathcal{F}$  contains all points in  $\bar{U} := U \setminus \{\mathbf{u}\}$ . We claim that  $\bar{U}$  is a spine of  $\bar{V} := V \cap \mathcal{F}$ . To see that, note that  $\mathcal{F}$  is spanned by  $\bar{V}$  and the facets of  $\mathcal{F}$  (considered as a polytope in  $\mathbb{R}^{d-1}$ ) are the ridges of  $\mathcal{P}$  contained in  $\mathcal{F}$ . Every such ridge  $\mathcal{R}$  is the intersection of  $\mathcal{F}$  and another facet  $\mathcal{F}'$  of  $\mathcal{P}$ . By assumption,  $\mathcal{F}'$  also contains at least  $n - 1$  points of  $U$  and it follows at once that  $\mathcal{R}$  contains  $n - 2$  points of  $\bar{U}$ . So,  $\bar{U}$  is a spine of  $\bar{V}$ , and by induction hypothesis, there exists a  $(d - 1)$ -simplex  $\bar{\sigma}$  in the  $\bar{U}$ -span in  $\bar{V}$  that contains  $\bar{\mathbf{p}}$ . The vertices of  $\bar{\sigma}$  together with  $\mathbf{u}$  span a simplex  $\sigma$  that contains  $\mathbf{p}$  and  $\sigma$  is in the  $U$ -span by construction.

The direction “ $\Leftarrow$ ” is clear if  $n \in \{0, 1\}$ , so we may assume that  $n \geq 2$  and proceed by contraposition. If  $U$  is not a spine, we have a facet  $\mathcal{F}$  of  $\mathcal{P}$  such that less than  $n - 1$  points of  $U$  lie on  $\mathcal{F}$ . Then, every simplex  $\sigma$  in the  $U$ -span has at least 2 vertices not on  $\mathcal{F}$ , and therefore at most  $d - 1$  vertices on  $\mathcal{F}$ . This implies that  $\sigma \cap \mathcal{F}$  is at most  $(d - 2)$ -dimensional. Therefore, the (finite) union of all  $U$ -span simplices cannot cover the  $(d - 1)$ -dimensional facet  $\mathcal{F}$ , which means that the  $U$ -span is not equal to  $\mathcal{P}$ .  $\square$

From now on, we use the (equivalent) geometric and combinatorial characterization of a spine without explicit reference to the preceding lemma. A useful property is that spines extend to faces in the following sense.

**Lemma 2.** *Let  $U$  be a spine of  $V$ , and let  $\mathcal{F}$  be an  $\ell$ -face of  $\mathcal{P}$ . Then  $\bar{U} := U \cap \mathcal{F}$  is a spine of  $\bar{V} := V \cap \mathcal{F}$ , both considered as point sets in  $\mathbb{R}^\ell$ . In particular,  $\mathcal{F}$  contains at least  $n - (d - \ell)$  points of  $U$ .*

*Proof.* For every simplex  $\sigma$  in the  $U$ -span, let  $\bar{\sigma} := \sigma \cap \mathcal{F}$ . Clearly,  $\bar{\sigma}$  is itself a simplex, spanned by the vertices  $V(\bar{\sigma}) = V(\sigma) \cap \mathcal{F}$ , and is of dimension at most  $\ell$ . Because  $U$  is a spine of  $V$ , the union of all  $\bar{\sigma}$  covers  $\mathcal{F} = \text{conv}(\bar{V})$ . Moreover,  $V(\bar{\sigma})$  contains  $\bar{U}$ . If  $\bar{\sigma}$  is not of dimension  $\ell$ , we can find an  $\ell$ -simplex in the  $\bar{U}$ -span of  $\bar{V}$  that has  $\bar{\sigma}$  as a face just by adding suitable vertices from  $\bar{V}$ . This implies that the union of the  $\bar{U}$ -span covers  $\mathcal{F}$ . The “in particular” part follows by induction on  $\ell$ .  $\square$

**Triangulations.** Let  $V' \subseteq \mathbb{R}^d$  be a finite point set that is full-dimensional, but not necessarily in convex position. We call a simplicial complex  $\mathcal{C}$  a *triangulation* of  $V'$  if  $V(\mathcal{C}) = V'$  and  $\bigcup \mathcal{C} = \text{conv}(V')$ . In this case, we also call  $\mathcal{C}$  a triangulation of the polytope  $\text{conv}(V')$ . In a triangulation of  $V'$ , every maximal simplex must be of dimension  $d$ .

We will consider two types of triangulations in this paper. For the first type, we assume that  $V' = V \cup \{\mathbf{0}\}$ , where  $\mathbf{0} = (0, \dots, 0) \notin V$ ,  $V$  is in convex position (as fixed before), and either  $\mathbf{0}$  lies in  $\text{conv}(V)$  or  $V'$  is in convex position as well. We define a *star triangulation* of  $V'$  as a triangulation where all  $d$ -simplices contain  $\mathbf{0}$  as a vertex (Figure 4).

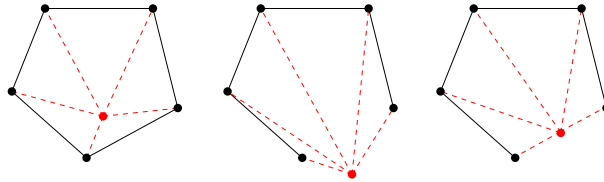


Figure 4: The three types of star triangulations.

**Lemma 3.** *A star triangulation of  $V'$  exists.*

*Proof.* We distinguish three cases, dependent on the position of  $\mathbf{0}$ . If it lies in the interior of  $\text{conv}(V)$ , we consider an arbitrary triangulation of  $V$  (for instance, the Delaunay triangulation [5, Ch. 9]), remove the  $d$ -simplices from this triangulation, and obtain a star triangulation by joining every  $(d-1)$ -simplex with  $\mathbf{0}$ .

If  $V'$  is in convex position, we proceed similarly: Starting with an arbitrary triangulation of  $V'$ , we remove the  $d$ -simplices to obtain a triangulation of the boundary facets, and join  $\mathbf{0}$  with every  $(d-1)$ -simplex that does not contain  $\mathbf{0}$  to obtain a star triangulation.

It remains the case that  $\mathbf{0}$  lies on the boundary of  $\text{conv}(V)$ . For this case, consider again a triangulation of  $V$ . Inductively in dimension, re-triangulate any  $k$ -face  $\tau$  that contains  $\mathbf{0}$  by joining  $\mathbf{0}$  with each  $(k-1)$ -face on the boundary of  $\tau$  that does not contain  $\mathbf{0}$ . The result of this process is a star triangulation (a conceptually simpler argument for the third case can be given by an analogue of a “pulling” perturbation of  $\mathbf{0}$ , as defined later in this section).  $\square$

If  $V$  is in simplicial position, there exists exactly one star triangulation of  $V' = V \cup \{\mathbf{0}\}$ .

Our second type of triangulations is defined for point sets in convex position. Fix a simplicial complex  $\mathcal{C}$  with vertex set  $V(\mathcal{C})$  in  $\mathbb{R}^d$  and a set  $U \subseteq V(\mathcal{C})$  of size at most  $d+1$ . Let  $\sigma$  denote the simplex spanned by  $U$ . We call  $\mathcal{C}$  *U-spinal* if every maximal simplex of  $\mathcal{C}$  contains  $\sigma$  as a face. If  $\mathcal{C}$  is a triangulation of  $V$ , we talk about a *U-spinal triangulation* accordingly. *U-spinal* triangulations are closely related to spines: if a *U-spinal* triangulation of  $V$  exists, then  $U$  is a spine of  $V$ , because all maximal simplices of the triangulation lie in the  $U$ -span. Moreover, if  $U$  consists of one element, say the origin, a  $\{\mathbf{0}\}$ -spinal triangulation is a star triangulation and therefore always exists by Lemma 3. For the previously discussed spine of a hypercube consisting of two opposite points, also a spinal triangulation exists, consisting of  $d!$   $d$ -simplices that all share the diagonal connecting these points. This construction is called *staircase triangulation* [7] or *Freudenthal triangulation* [8]. Our second main result (Main Theorem 2, the general lifting theorem) shows that  $U$  being a spine is not only a necessary but also a sufficient condition for a *U-spinal* triangulation to exist.

**Perturbations.** We call a set  $\tilde{V} \subseteq \mathbb{R}^d$  a  $\delta$ -perturbation of  $V$  with  $\delta > 0$  if there is a one-to-one correspondence of points in  $V$  and  $\tilde{V}$  such that the distance of corresponding points is at most  $\delta$ . The following statements are well-known:

**Theorem 4.**

- Every  $\delta$ -perturbation  $\tilde{V}$  with  $\delta$  sufficiently small is full-dimensional and in convex position.
- For arbitrary  $\delta > 0$  a  $\delta$ -perturbation  $\tilde{V}$  with each perturbed point chosen uniformly at random is in simplicial position with probability 1.

Theorem 4 shows that non-simplicial point configurations form a degenerate set among all possible point configurations. For that reason, simplicial position is also called *general position* sometimes. It is common practice to assume simplicial position of  $V$  without loss of generality, passing to a  $\delta$ -perturbation if needed. For our results, however, it will turn out that a random perturbation will not suffice, and we have to devise a more “controlled” perturbation scheme. This requires the study of the effect of perturbing a vertex of a polytope in some more detail.

In the following, we fix a point  $\mathbf{v} \in V$ , and we let  $\tilde{\mathbf{v}}$  denote a point in distance  $\delta$  to  $\mathbf{v}$ , where  $\delta > 0$  is sufficiently small. We let  $\tilde{V}$  denote the point set  $V$  with  $\mathbf{v}$  replaced by  $\tilde{\mathbf{v}}$ . Note that  $\tilde{V}$  is a special case of a  $\delta$ -perturbation where all vertices except one remain fixed. We call this an *elementary  $\delta$ -perturbation*. Letting  $\tilde{\mathcal{P}}$  denote the polytope spanned by  $\tilde{V}$ , we are interested in the differences of the facet structures of  $\mathcal{P}$  and  $\tilde{\mathcal{P}}$  after applying an elementary  $\delta$ -perturbation. We will often identify a facet with its list of vertices in the following arguments.

**Lemma 5.** *If  $\delta$  is sufficiently small, the facets of  $\tilde{\mathcal{P}}$  that do not contain  $\tilde{\mathbf{v}}$  are the same as the facets of  $\mathcal{P}$  that do not contain  $\mathbf{v}$ . Moreover, if a set of points  $\{\tilde{\mathbf{v}}, \mathbf{v}_1, \dots, \mathbf{v}_k\}$  spans a facet of  $\tilde{\mathcal{P}}$ , then the points  $\{\mathbf{v}, \mathbf{v}_1, \dots, \mathbf{v}_k\}$  lie on a common facet of  $\mathcal{P}$ .*

*Proof.* For the first part, let  $\mathcal{F}$  be a facet of  $\mathcal{P}$ , and let  $\mathbb{H}$  be the support of  $\mathcal{F}$ . If  $\mathbf{v} \notin \mathcal{F}$ , then  $\mathbb{H}$  is still a supporting hyperplane for  $\tilde{\mathcal{P}}$  (if  $\delta$  is sufficiently small), thus  $\mathcal{F}$  is a facet of  $\tilde{\mathcal{P}}$  as well. For the second part, assume to the contrary that the convex hull of  $\{\mathbf{v}, \mathbf{v}_1, \dots, \mathbf{v}_k\}$  does not lie on a facet of  $\mathcal{P}$ , so it passes

through the interior of  $\mathcal{P}$ . A sufficiently small perturbation then ensures that  $\{\tilde{\mathbf{v}}, \mathbf{v}_1, \dots, \mathbf{v}_k\}$  also lies in the interior of  $\tilde{\mathcal{P}}$ , which is a contradiction.  $\square$

Lemma 5 shows that the effect of perturbing  $\mathbf{v}$  is that facets containing  $\mathbf{v}$  are split into sub-facets (with  $\mathbf{v}$  replaced by  $\tilde{\mathbf{v}}$ ). It makes therefore sense to say that a facet in  $\tilde{\mathcal{P}}$  *originates* from a unique facet in  $\mathcal{P}$ . We analyze this splitting procedure in more detail: let  $\mathcal{F}$  be a facet of  $\mathcal{P}$  with  $\mathbf{v}$  as vertex, and let  $\mathbb{H}$  be the support of  $\mathcal{F}$ . Let  $\mathbb{H}^-$  denote the open half-space defined by  $\mathbb{H}$  which contains the interior of  $\mathcal{P}$ , and  $\mathbb{H}^+$  the complementary open half-space. We call the elementary  $\delta$ -perturbation  $\tilde{V}$  a *push of  $\mathbf{v}$  with respect to  $\mathcal{F}$*  if  $\tilde{\mathbf{v}}$  lies in  $\mathbb{H}^-$  and a *pull of  $\mathbf{v}$  with respect to  $\mathcal{F}$*  if it lies in  $\mathbb{H}^+$  (we ignore the degenerate case that  $\tilde{\mathbf{v}}$  still lies on  $\mathbb{H}$ ).

The next two lemmas describe the facet structure after a pull and a push. Both statements are proved under the name of “subdivision of point configurations” in [7, Lemma 4.3.10]. We include proof outlines adapted to our notation for convenience.

**Lemma 6.** *Let  $\mathbf{v}, \mathbf{v}_1, \dots, \mathbf{v}_k$  be distinct vertices of  $\mathcal{F}$ . If  $\tilde{V}$  is a pull of  $\mathbf{v}$  with respect to  $\mathcal{F}$ ,  $\{\tilde{\mathbf{v}}, \mathbf{v}_1, \dots, \mathbf{v}_k\}$  spans a facet of  $\tilde{\mathcal{P}}$  if and only if  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  spans a ridge of  $\mathcal{P}$ .*

*Proof.* We assume without loss of generality that the support  $\mathbb{H}$  of  $\mathcal{F}$  is the hyperplane defined by  $x_d = 0$ , and that the polytope is contained in the half-space  $x_d \leq 0$ . Because we assume a pull, the  $x_d$ -coordinate of  $\tilde{\mathbf{v}}$  is positive. Assume that  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  spans a ridge  $\mathcal{R}$  of  $\mathcal{P}$ . Considering  $\mathcal{F}$  as a polytope in  $\mathbb{R}^{d-1}$ ,  $\mathcal{R}$  is a facet of  $\mathcal{F}$ , and  $\mathcal{F}$  lies in one of the closed half-spaces of  $\mathbb{R}^{d-1}$  induced by  $\mathcal{R}$ . Going back to  $\mathbb{R}^d$ ,  $\mathcal{R}$  and  $\tilde{\mathbf{v}}$  span a hyperplane  $\tilde{\mathbb{H}}$  which intersects  $\mathcal{F}$  in the ridge  $\mathcal{R}$ . We say that a point  $\mathbf{x} \in \mathbb{R}^d \setminus \tilde{\mathbb{H}}$  lies *below*  $\tilde{\mathbb{H}}$  if the vertically upward ray from  $\mathbf{x}$  in  $x_d$ -direction intersects  $\tilde{\mathbb{H}}$ . Each vertex of  $\tilde{\mathcal{P}}$  lies on or below  $\tilde{\mathbb{H}}$ . This is clear for vertices not on  $\mathcal{F}$  if the perturbation is sufficiently small. Also  $\tilde{\mathbf{v}}$  and all vertices on  $\mathcal{R}$  lie on  $\tilde{\mathbb{H}}$  by construction. It is not hard to show that every vertex on  $\mathcal{F} \setminus \mathcal{R}$  lies below  $\tilde{\mathbb{H}}$  as well, because  $\mathbf{v}$  lies below by construction, and any other vertex is on the same side as  $\mathbf{v}$  with respect to  $\mathcal{R}$ .

Since every vertex of  $\tilde{\mathcal{P}}$  lies on or below  $\tilde{\mathbb{H}}$ ,  $\tilde{\mathbb{H}}$  is a supporting hyperplane for  $\tilde{\mathcal{P}}$ , which means that  $\{\tilde{\mathbf{v}}, \mathbf{v}_1, \dots, \mathbf{v}_k\}$  spans a facet. This shows that all such ridges of  $\mathcal{P}$  indeed induce facets of  $\tilde{\mathcal{P}}$ . To see that there are not more facets, we consider the projection of the facets (constructed in the first part) to  $\mathbb{H}$ , just by replacing  $\tilde{\mathbf{v}}$  by  $\mathbf{v}$ . It is not difficult to show that the union of the projected facets covers  $\mathcal{F}$ , which implies that there cannot be any additional facet originating from  $\mathcal{F}$ .  $\square$

For the push case, we need the following definition: for a face  $\mathcal{F}$  of a (not necessarily full-dimensional) polytope  $\mathcal{Q}$ , we say that  $\mathcal{F}$  is *visible* from a point  $\mathbf{x} \in \mathbb{R}^d$ , if for all points  $\mathbf{p} \in \mathcal{F}$ , the line segment  $\mathbf{x}\mathbf{p}$  intersects  $\mathcal{Q}$  exactly in the point  $\mathbf{p}$ . If  $\mathcal{F}$  is a facet of  $\mathcal{Q}$ , an equivalent definition is that the support  $\mathbb{H}$  of  $\mathcal{F}$  separates  $\mathbf{x}$  and  $\mathcal{Q}$  (that is,  $\mathbf{x}$  lies in an open half-space defined by  $\mathbb{H}$ , and  $\mathcal{Q} \setminus \mathcal{F}$  lies in the opposite open half-space); see [7, Lemma 4.3.1] for the (simple) proof. In particular, a facet  $\mathcal{F}$  of  $\mathcal{Q}$  is not visible from  $\mathbf{x}$  if  $\mathbf{x}$  lies on the support of  $\mathcal{F}$ .

**Lemma 7.** *Let  $\tilde{V}$  be a push of  $\mathbf{v}$  with respect to  $\mathcal{F}$  and let  $\mathcal{F}_{-\mathbf{v}}$  be the polytope spanned by  $V(\mathcal{F}) \setminus \{\mathbf{v}\}$ . If  $\mathcal{F}_{-\mathbf{v}}$  is  $(d-1)$ -dimensional, it forms a facet of  $\tilde{\mathcal{P}}$ . Moreover,  $\{\tilde{\mathbf{v}}, \mathbf{v}_1, \dots, \mathbf{v}_k\}$  with  $\mathbf{v}_1, \dots, \mathbf{v}_k \in V(\mathcal{F}) \setminus \{\mathbf{v}\}$  spans a facet of  $\tilde{\mathcal{P}}$  if and only if  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  spans a  $(d-2)$ -face of  $\mathcal{F}_{-\mathbf{v}}$  that is visible from  $\mathbf{v}$ .*

*Proof.* Similar as in the proof of Lemma 6, assume that the support  $\mathbb{H}$  of  $\mathcal{F}$  is defined by  $x_d = 0$ , and the polytope lies in the half-space  $x_d \leq 0$ . The perturbed point  $\tilde{\mathbf{v}}$  has negative  $x_d$ -coordinate, because we assume a push.  $\mathbb{H}$  remains supporting after the perturbation, so  $\mathcal{F}_{-\mathbf{v}}$  is a face of  $\tilde{\mathcal{P}}$ . Therefore, it is a facet if and only if it is  $(d-1)$ -dimensional.

Let  $\mathcal{R} := \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  denote a  $(d-2)$ -face of  $\mathcal{F}_{-\mathbf{v}}$  that is visible from  $\mathbf{v}$ . Note that  $\mathbf{v}$ ,  $\mathcal{F}_{-\mathbf{v}}$ , and  $\mathcal{R}$  lie on  $\mathbb{H}$ , which we identify with  $\mathbb{R}^{d-1}$ . In  $\mathbb{H}$ , by definition of visibility, the support of  $\mathcal{R}$  (being a space of dimension  $d-2$ ) separates  $\mathcal{F}_{-\mathbf{v}}$  and  $\mathbf{v}$ . Going back to  $\mathbb{R}^d$ , let  $\tilde{\mathbb{H}}$  denote the hyperplane spanned by  $\{\tilde{\mathbf{v}}, \mathbf{v}_1, \dots, \mathbf{v}_k\}$ . The unperturbed point  $\mathbf{v}$  lies above  $\tilde{\mathbb{H}}$ . (in the same sense as in the proof of Lemma 6). For any vertex  $\mathbf{w}$  on  $\mathcal{F}_{-\mathbf{v}}$ , the line segment connecting  $\mathbf{w}$  and  $\mathbf{v}$  intersects  $\mathcal{R}$ , because of its separation property. This implies that  $\mathbf{w}$  lies on or below  $\tilde{\mathbb{H}}$ . It follows that  $\tilde{\mathbb{H}}$  is a supporting hyperplane for  $\tilde{\mathcal{P}}$  and  $\{\tilde{\mathbf{v}}, \mathbf{v}_1, \dots, \mathbf{v}_k\}$  is a facet. Similar as in the proof of Lemma 6, projecting the obtained facets back to  $\mathcal{F}$ , we observe that  $\mathcal{F}$  is covered, implying that there is no additional facet originating from  $\mathcal{F}$ .  $\square$

Let  $\mathcal{F}$  be a facet of  $\mathcal{P}$  containing a vertex  $\mathbf{v}$ . We call  $\mathbf{v}$  a *tip* of  $\mathcal{F}$  if the remaining vertices of  $\mathcal{F}$  lie in a common ridge of  $\mathcal{P}$  (which necessarily does not contain  $\mathbf{v}$ ). Equivalently,  $\mathbf{v}$  is a tip of  $\mathcal{F}$  if and only if the affine span of  $V(\mathcal{F}) \setminus \{\mathbf{v}\}$  does not contain  $\mathbf{v}$ . A facet is a simplex if and only if each vertex is a tip. Also, if  $\mathbf{v}$  is a tip of  $\mathcal{F}$ , it can be readily verified that after a push or a pull, the facet remains a facet of  $\tilde{\mathcal{P}}$  with  $\mathbf{v}$  replaced by  $\tilde{\mathbf{v}}$ . The next lemma shows that in general, the polytope gathers more tips when applying a pull or a push, and therefore gets “closer to simplicial”.

**Lemma 8.** *If  $\tilde{V}$  is a push or a pull of  $\mathbf{v}$  with respect to  $\mathcal{F}$ , the perturbed point  $\tilde{\mathbf{v}}$  is a tip of all facets that originate from  $\mathcal{F}$  and contain  $\tilde{\mathbf{v}}$ . Moreover, if a vertex  $\mathbf{w} \in V(\mathcal{F}) \setminus \{\mathbf{v}\}$  is a tip of  $\mathcal{F}$ ,  $\mathbf{w}$  is also a tip of every facet that originates from  $\mathcal{F}$  and contains  $\mathbf{w}$ .*

*Proof.* Let  $\tilde{\mathcal{F}}$  be a facet of  $\tilde{\mathcal{P}}$  that originates from  $\mathcal{F}$ . The first part follows from Lemma 6 and Lemma 7, just by observing that all facets containing  $\tilde{\mathbf{v}}$  are formed by joining  $\tilde{\mathbf{v}}$  with a  $(d-2)$ -dimensional object.

The second part is obvious if  $\tilde{\mathbf{v}}$  does not lie on  $\tilde{\mathcal{F}}$ , so we are left with the case that the vertices of  $\tilde{\mathcal{F}}$  are  $\{\mathbf{w}, \tilde{\mathbf{v}}, \mathbf{v}_1, \dots, \mathbf{v}_k\}$  with some  $\mathbf{v}_1, \dots, \mathbf{v}_k \in V(\mathcal{F})$ . By assumption,  $\mathbf{w}$  is a tip of  $\mathcal{F}$ , so the points  $\{\mathbf{v}, \mathbf{v}_1, \dots, \mathbf{v}_k\}$  are contained in a ridge  $\mathcal{R}$  of  $\mathcal{P}$  such that  $\mathbf{w} \notin \mathbb{A}$  where  $\mathbb{A}$  is the affine span of  $\mathcal{R}$ . The perturbed point  $\tilde{\mathbf{v}}$  together with  $\mathbb{A}$  spans a hyperplane  $\tilde{\mathbb{H}}$  because  $\mathbf{v}$  is perturbed out of the support  $\mathbb{H}$  of  $\mathcal{F}$  and  $\mathbb{A} \subseteq \mathbb{H}$ .  $\tilde{\mathbb{H}}$  and  $\mathbb{H}$  both contain  $\mathbb{A}$  but are not equal since  $\tilde{\mathbf{v}} \in \tilde{\mathbb{H}} \setminus \mathbb{H}$ , hence  $\tilde{\mathbb{H}} \cap \mathbb{H} = \mathbb{A}$ . In particular,  $\mathbf{w}$  (which is in  $\mathbb{H} \setminus \mathbb{A}$ ) does not lie on  $\tilde{\mathbb{H}}$ . It follows that the affine span of  $\{\tilde{\mathbf{v}}, \mathbf{v}_1, \dots, \mathbf{v}_k\}$  (which is a subset of  $\tilde{\mathbb{H}}$ ) does not contain  $\mathbf{w}$ , which implies that  $\mathbf{w}$  is a tip of  $\tilde{\mathcal{F}}$ .  $\square$

We will also make use of the following simple observation which characterizes the facets obtained by a push.

**Lemma 9.** *Let  $\mathcal{F}$  be a facet of  $\mathcal{P}$  and  $\mathbf{v}$  be a vertex of  $\mathcal{F}$ . Let  $\mathcal{F}_{-\mathbf{v}}$  be the polytope spanned by  $V(\mathcal{F}) \setminus \{\mathbf{v}\}$  and let  $\mathbf{w} \in V(\mathcal{F}) \setminus \{\mathbf{v}\}$  be a tip of  $\mathcal{F}$ . Then, no  $(d-2)$ -face of  $\mathcal{F}_{-\mathbf{v}}$  that does not contain  $\mathbf{w}$  is visible from  $\mathbf{v}$ . In particular, after a push of  $\mathbf{v}$  with respect to  $\mathcal{F}$ , all facets originating from  $\mathcal{F}$  contain  $\mathbf{w}$  as vertex.*

*Proof.* For the first statement, we can assume that  $\mathcal{F}_{-\mathbf{v}}$  is of dimension  $(d-1)$  because otherwise, no  $(d-2)$ -face not containing  $\mathbf{w}$  exists and the statement is trivial. Because  $\mathbf{w}$  is a tip of  $\mathcal{F}$ , all points in  $V(\mathcal{F}) \setminus \{\mathbf{w}\}$  lie on a ridge  $\mathcal{R}$  of  $\mathcal{P}$ . This ridge is a facet of  $\mathcal{F}$ , considered as a polytope in  $\mathbb{R}^{d-1}$ . Let  $\mathbb{H}$  denote its support in  $\mathbb{R}^{d-1}$ . Now consider a  $(d-2)$ -face  $\mathcal{R}'$  of  $\mathcal{F}_{-\mathbf{v}}$  not containing  $\mathbf{w}$ . Because we assume that  $\mathcal{F}_{-\mathbf{v}}$  is a  $(d-1)$ -dimensional polytope,  $\mathcal{R}'$  is a facet of  $\mathcal{F}_{-\mathbf{v}}$ , and it is contained in  $\mathcal{R}$  by construction. Thus  $\mathbb{H}$  is also the support of  $\mathcal{R}'$ . If  $\mathcal{R}'$  is visible from  $\mathbf{v}$ ,  $\mathbb{H}$  has to separate  $\mathbf{v}$  from  $\mathcal{F}_{-\mathbf{v}} \setminus \mathcal{R}'$ . But this is impossible, because  $\mathbf{v} \in \mathbb{H}$ . The second statement follows directly from the first part and Lemma 7.  $\square$

So far, we only defined a push and a pull of  $\mathbf{v}$  with respect to a given facet. More generally, we call the elementary  $\delta$ -perturbation  $\tilde{V}$  of  $V$  a *push* (*pull*) of  $\mathbf{v}$  if it is a push (pull) of  $\mathbf{v}$  with respect to every facet of  $\mathcal{P}$  that contains  $\mathbf{v}$ . Indeed, note that  $\mathbf{v}$  is “pushed into” or “pulled from”  $\mathcal{P}$  under the perturbation. In general a randomly chosen perturbation is neither a push nor a pull as it can be of mixed type, being a push for some facets and a pull for others. Nevertheless, the space of push and pull  $\delta$ -perturbations is the intersection of a  $\delta$ -ball around  $\mathbf{v}$  with finitely many half-spaces, and therefore of positive volume.

The results of this section can be combined to prove the following theorem which essentially states that if  $V$  (which is in convex but not necessarily simplicial position) has a spine  $U$  then there is a  $\delta$ -perturbation  $\tilde{V}$  of  $V$  (a composition of sufficiently small pulls and pushes) which is in convex and simplicial position and the set  $\tilde{U}$  of perturbed points in  $U$  is still a spine of  $\tilde{V}$ . This is the version of Theorem 4 that fits our purposes.

**Theorem 10** (Transformation to simplicial position). *Let  $U = \{\mathbf{u}_1, \dots, \mathbf{u}_n\} \subseteq V$  be a (strong) spine of  $V$ ,  $m := |V|$ , and  $V \setminus U = \{\mathbf{v}_{n+1}, \dots, \mathbf{v}_m\}$ . Furthermore let  $\tilde{V}_0, \dots, \tilde{V}_m \subseteq \mathbb{R}^d$  such that  $\tilde{V}_0 = V$  and  $\tilde{V}_i$  is an elementary  $\delta_i$  perturbation of  $\tilde{V}_{i-1}$  with  $\delta_i$  sufficiently small that is a pull of  $\mathbf{u}_i$  for  $i \in \{1, \dots, n\}$  and a push of  $\mathbf{v}_i$  for  $i \in \{n+1, \dots, m\}$ , yielding the point  $\tilde{\mathbf{u}}_i$  or  $\tilde{\mathbf{v}}_i$ , respectively. With  $\delta := \max\{\delta_1, \dots, \delta_m\}$ ,  $\tilde{U} := \{\tilde{\mathbf{u}}_1, \dots, \tilde{\mathbf{u}}_n\}$  is a  $\delta$ -perturbation of  $U$ ,  $\tilde{V} := \tilde{V}_m$  is a  $\delta$ -perturbation of  $V$ ,  $\tilde{V}$  is in convex and simplicial position, and  $\tilde{U}$  is a (strong) spine of  $\tilde{V}$ .*

*Proof.* To reduce notation we will identify a polytope with its list of vertices. Obviously,  $\tilde{U}$  and  $\tilde{V}$  are  $\delta$ -perturbations of  $U$  and  $V$ , respectively. Since  $\tilde{V}$  is a composition of pulls and pushes of all points in  $V$ ,



it follows from Lemma 8 that every point in  $\tilde{V}$  is a tip of all facets of  $\tilde{V}$  the point is contained in, hence all facets of  $\tilde{V}$  are simplices, i.e.  $\tilde{V}$  is in simplicial position.

We are left to show that  $\tilde{U}$  is a spine of  $\tilde{V}$ . First we observe that since  $U$  is a spine of  $V$ , all ridges of  $V$  contain at least  $n - 2$  points of  $U$  by Lemma 2. By Lemma 5 and Lemma 6, all facets of  $\tilde{V}_1$  (which is a pull of  $\mathbf{u}_1$ ) contain at least  $n - 1$  points of  $\{\tilde{\mathbf{u}}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ , which is thus a spine of  $\tilde{V}_1$ . By induction, it follows that  $\{\tilde{\mathbf{u}}_1, \dots, \tilde{\mathbf{u}}_i, \mathbf{u}_{i+1}, \dots, \mathbf{u}_n\}$  is a spine of  $\tilde{V}_i$  for all  $i \in \{1, \dots, n\}$ . In particular,  $\tilde{U}$  is a spine of  $\tilde{V}_n$ . Lemma 8 implies that all points in  $\tilde{U}$  are tips of all facets of  $\tilde{V}_n$  they are contained in. But then it follows from Lemma 9 that all facets of  $\tilde{V}$  also contain at least  $n - 1$  points of  $\tilde{U}$  (the points in  $V \setminus U$  are all “pushed”). Thus  $\tilde{U}$  is a spine of  $\tilde{V}$ . Furthermore, if  $U$  is a strong spine of  $V$ , Lemma 5 asserts that no facet of  $\tilde{\mathcal{P}}$  can contain  $n$  points of  $U$ , which implies that  $\tilde{U}$  is a strong spine of  $\tilde{V}$ .  $\square$

**Rotations of hyperplanes.** We review some basic facts about supporting hyperplanes in  $\mathbb{R}^d$  and the process of rotating them around a  $(d - 2)$ -dimensional axis. For all the basic results obtained in this section, it is instructive to visualize them in  $\mathbb{R}^3$ .

In this section, we assume  $V$  to be in simplicial position, so that all facets of  $\mathcal{P}$  are simplices. We consider a facet  $\mathcal{F}$  of  $\mathcal{P}$  and let  $\mathbb{H}$  denote its support. We let  $\mathbb{H}^-$  denote the open halfspace defined by  $\mathbb{H}$  which contains the interior of  $\mathcal{P}$ , and  $\mathbb{H}^+$  the opposite open halfspace. Considering  $\mathcal{F}$  as a polytope in  $\mathbb{R}^{d-1}$ , let  $\mathbb{A}$  denote a supporting hyperplane in  $\mathbb{R}^{d-1}$  for the polytope  $\mathcal{F}$  (that is,  $\mathbb{A}$  is of dimension  $(d - 2)$ ). Assume without loss of generality that  $\mathbb{A}$  contains the origin, and let  $\mathbf{b}_1, \dots, \mathbf{b}_{d-2}$  be an orthonormal basis of  $\mathbb{A}$ . Let  $\mathbf{b}_{d-1}$  be the unit normal vector of  $\mathbb{A}$  in  $\mathbb{H}$  pointing towards  $\mathcal{F}$ , and  $\mathbf{b}_d$  be the unit normal vector of  $\mathbb{H}$  pointing into  $\mathbb{H}^+$ . We define the  $\alpha$ -rotation of  $\mathbb{H}$  around  $\mathbb{A}$  (with respect to  $\mathcal{P}$ ) as the hyperplane in  $\mathbb{R}^d$  spanned by  $\mathbf{b}_1, \dots, \mathbf{b}_{d-2}, \cos(\alpha)\mathbf{b}_{d-1} + \sin(\alpha)\mathbf{b}_d$ . In other words, this is the hyperplane obtained by rotating  $\mathbb{H}$  around  $\mathbb{A}$  by an angle of  $\alpha$ . The following property is straight-forward to check, and we omit its proof.

**Lemma 11.** *There exists an  $\epsilon > 0$  such that the  $\alpha$ -rotation of  $\mathbb{H}$  is a supporting hyperplane for  $\mathcal{P}$  for  $\alpha \in [0, \epsilon]$ , and not a supporting hyperplane for  $\mathcal{P}$  for  $\alpha \in [-\epsilon, 0)$ .*

For every vertex  $\mathbf{v}$  of  $\mathcal{P}$  that does not lie on  $\mathbb{A}$ , there exists a unique  $\alpha_{\mathbf{v}} \in (0, \pi]$  such that  $\mathbf{v}$  lies on the  $\alpha_{\mathbf{v}}$ -rotation of  $\mathbb{H}$ . These  $\alpha_{\mathbf{v}}$ -values induce an ordering of the vertices not on  $\mathbb{A}$ . Considering the rotation as a continuous process, the rotated hyperplane is supporting initially by Lemma 11 and stays supporting as long as no vertex of  $\mathcal{P}$  is hit. The next statement follows immediately (exploiting the simplicial position of  $V$ ).

**Lemma 12.** *Let  $\mathbf{v}_1, \dots, \mathbf{v}_k$  denote the vertices of  $\mathcal{P}$  on  $\mathbb{A}$ , and let  $\mathbf{v}$  denote a vertex not on  $\mathbb{A}$  with minimal  $\alpha$ -value. Then,  $\mathbf{v}, \mathbf{v}_1, \dots, \mathbf{v}_k$  lie on a common face of  $\mathcal{P}$ .*

Let  $\mathbb{G}$  be another hyperplane in  $\mathbb{R}^d$  not parallel to  $\mathbb{H}$  such that  $\mathbb{A} = \mathbb{H} \cap \mathbb{G}$ . Then, there exists an angle  $\alpha_{\mathbb{G}} \in (0, \pi)$  such that  $\mathbb{G}$  is the  $\alpha_{\mathbb{G}}$ -rotation of  $\mathbb{H}$ . Moreover, we denote the open halfspace  $\mathbb{G}^-$  such that the  $\alpha_{\mathbb{G}}$ -rotation transforms  $\mathbb{H}^-$  into  $\mathbb{G}^-$ , and we let  $\mathbb{G}^+$  denote the opposite open halfspace.

**Lemma 13.** *Let  $\mathbf{v} \in V$  be in  $\mathbb{G}^+$  and not on  $\mathcal{F}$ . Then  $\alpha_{\mathbf{v}} < \alpha_{\mathbb{G}}$ .*

*Proof.* Consider the continuous process of transforming  $\mathbb{H}^-$  into  $\mathbb{G}^-$  by rotation. Since  $\mathbf{v}$  lies in  $\mathbb{H}^-$  (as it is not in  $\mathcal{F}$ ) and not in  $\mathbb{G}^-$ , there must be some hyperplane  $\mathbb{H}'$  in-between such that  $\mathbf{v} \in \mathbb{H}'$ .  $\mathbb{H}'$  is an  $\alpha$ -rotation with  $0 < \alpha < \alpha_{\mathbb{G}}$  and  $\alpha_{\mathbf{v}} = \alpha$  by definition.  $\square$

### 3 The Everest polytope

For  $n, s \in \mathbb{N}$ , define  $\mathcal{E}_{n,s} := \{\mathbf{x} \in \mathbb{R}^{ns} \mid g_{n,s}(\mathbf{x}) \leq 1\}$  where  $g_{n,s} : \mathbb{R}^{ns} \rightarrow \mathbb{R}$  with

$$(x_{1,1}, \dots, x_{n,s}) \mapsto \sum_{j=1}^s \max \{0, x_{1,j}, \dots, x_{n,j}\} + \max \left\{ 0, -\sum_{j=1}^s x_{1,j}, \dots, -\sum_{j=1}^s x_{n,j} \right\}.$$

It is not difficult to verify that  $\mathcal{E}_{n,s}$  is bounded and the intersection of finitely many halfspaces of  $\mathbb{R}^{ns}$ . We call it the  $(n, s)$ -Everest polytope. It is well-known [2] that the number-theoretic constant  $c(n, s)$  discussed in the introduction is equal to the volume of  $\mathcal{E}_{n,s}$ .

**Vertex sets.** In order to describe the vertices of  $\mathcal{E}_{n,s}$  we introduce the following point sets which we also utilize in later parts of the paper. Note that we identify  $\mathbb{R}^{ns}$  and  $\mathbb{R}^{n \times s}$  which explains the meaning of “row” and “column” in the definition. Let  $\mathbf{e}_s(i)$  denote the  $i$ -th  $s$ -dimensional unit (row) vector with the convention that  $\mathbf{e}_s(0) = \mathbf{0}$ . We define the following sets of points in  $\mathbb{R}^{ns}$ :

$$\begin{aligned} V_{n,s} &:= \left\{ \begin{pmatrix} -\mathbf{e}_s(j_1) \\ \vdots \\ -\mathbf{e}_s(j_n) \end{pmatrix} \mid j_1, \dots, j_n \in \{0, \dots, s\} \right\}, \\ U_{n,s} &:= \left\{ \begin{pmatrix} -\mathbf{e}_s(j) \\ \vdots \\ -\mathbf{e}_s(j) \end{pmatrix} \mid j \in \{0, \dots, s\} \right\}, \\ P_{n,s} &:= V_{n,s} - (U_{n,s} \setminus \{\mathbf{0}\}) = \{\mathbf{v} - \mathbf{u} \mid \mathbf{v} \in V_{n,s} \wedge \mathbf{u} \in U_{n,s} \setminus \{\mathbf{0}\}\}. \end{aligned}$$

It can be readily verified that  $V_{n,s}$  is the set of points in  $\{-1, 0\}^{ns}$  such that there is at most one  $-1$  per row,  $U_{n,s}$  is the set of points in  $V_{n,s}$  such that all  $-1$ 's (if there are any) are contained in a single column, and  $P_{n,s}$  is the set of points in  $\{-1, 0, 1\}^{ns}$  such that

- all '1's (if there are any) are in a unique "1-column",
- all entries of the 1-column are either 0 or 1,
- all rows with a 1 contain at most one  $-1$ ,
- all rows without a 1 contain only '0's.

**Lemma 14.**  $P_{n,s} \cap V_{n,s} = \{\mathbf{0}\}$ ,  $U_{n,s} \subseteq V_{n,s}$ ,  $|V_{n,s}| = (s+1)^n$ ,  $|U_{n,s}| = s+1$ , and  $|P_{n,s}| = s(s+1)^n - s + 1$ .

*Proof.* Follows directly from the definitions and from basic combinatorics.  $\square$

**Theorem 15.** The set of vertices of  $\mathcal{E}_{n,s}$  is given by  $(P_{n,s} \cup V_{n,s}) \setminus \{\mathbf{0}\} = (V_{n,s} - U_{n,s}) \setminus \{\mathbf{0}\}$ .

We split the proof into several parts which will be treated in the following lemmas. For the rest of this section, let  $i$  and  $j$  (with a possible subscript) denote elements of  $\{1, \dots, n\}$  and  $\{1, \dots, s\}$ , respectively, let  $\mathbf{v} = (v_{1,1}, \dots, v_{n,s})$  be a vertex of  $\mathcal{E}_{n,s}$ , and set

$$\begin{aligned} m_j &:= \max\{0, v_{1,j}, \dots, v_{n,j}\} \text{ for all } j, \\ s_i &:= -\sum_{j=1}^s v_{i,j} \text{ for all } i, \\ m &:= \max\{0, s_1, \dots, s_n\}. \end{aligned}$$

Then

$$\begin{aligned} g_{n,s} \begin{pmatrix} v_{1,1} & \cdots & v_{1,s} \\ \vdots & & \vdots \\ v_{n,1} & \cdots & v_{n,s} \end{pmatrix} &= \max \begin{pmatrix} 0 \\ v_{1,1} \\ \vdots \\ v_{n,1} \end{pmatrix} + \cdots + \max \begin{pmatrix} 0 \\ v_{1,s} \\ \vdots \\ v_{n,s} \end{pmatrix} + \max \begin{pmatrix} 0 \\ -v_{1,1} - \cdots - v_{1,s} \\ \vdots \\ -v_{n,1} - \cdots - v_{n,s} \end{pmatrix} \\ &= m_1 + \cdots + m_s + m \\ &= 1. \end{aligned}$$

Furthermore it can easily be verified that  $v_{i,j} \in [-1, 1]$  for all  $i$  and  $j$ ,  $m_j \in [0, 1]$  for all  $j$ ,  $s_i \in [-1, 1]$  for all  $i$ , and  $m \in [0, 1]$ .

In the proofs below, we will repeatedly apply the following argument: if there is an  $\varepsilon > 0$  and an  $\mathbf{x} \in \mathbb{R}^{ns}$  such that  $\mathbf{v} \pm \delta \mathbf{x} \in \mathcal{E}_{n,s}$  for all  $\delta \in [0, \varepsilon]$ ,  $\mathbf{v}$  cannot be a vertex of  $\mathcal{E}_{n,s}$  (since it is in the interior of an at least 1-dimensional face).

**Lemma 16.** If  $m_j = 0$  for all  $j$  then  $\mathbf{v} \in V_{n,s}$ .

*Proof.* Since all  $m_j$  are equal to zero, all  $v_{i,j}$  have to be non-positive, so all  $s_i$  are non-negative. Also we get that  $m$  is equal to 1 which implies that at least one of the  $s_i$  is equal to 1. Suppose that  $v_{i_0,j_0} \in (-1, 0)$  for some  $i_0, j_0$ . If  $s_{i_0} < 1$ , then for  $\varepsilon := \min\{-v_{i_0,j_0}, 1 - s_{i_0}\} > 0$  and  $\delta \in [0, \varepsilon]$  we get that  $g_{n,s}(v_{1,1}, \dots, v_{i_0,j_0} \pm \delta, \dots, v_{n,s}) = 1$ , so  $(v_{1,1}, \dots, v_{i_0,j_0} \pm \delta, \dots, v_{n,s})$  is on the boundary of  $\mathcal{E}_{n,s}$  and  $\mathbf{v}$  cannot be a vertex of  $\mathcal{E}_{n,s}$ . To see this more easily consider the example

$$\mathbf{v} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1/6 & 0 & -2/3 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1/6 \pm \delta & 0 & -2/3 & 0 \end{pmatrix}.$$

If on the other hand  $s_{i_0}$  is equal to 1, then there is a  $j_1 \neq j_0$  such that  $v_{i_0,j_1} \in (-1, 0)$ . But then we get  $g_{n,s}(v_{1,1}, \dots, v_{i_0,j_0} \pm \delta, \dots, v_{i_0,j_1} \mp \delta, \dots, v_{n,s}) = 1$  for  $\varepsilon := \min\{-v_{i_0,j_0}, -v_{i_0,j_1}, v_{i_0,j_0} + 1, v_{i_0,j_1} + 1\} > 0$  and  $\delta \in [0, \varepsilon]$ , so again  $\mathbf{v}$  cannot be a vertex of  $\mathcal{E}_{n,s}$ . Again we give an example

$$\mathbf{v} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1/6 & -1/6 & 0 & -2/3 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1/6 \pm \delta & -1/6 \mp \delta & 0 & -2/3 \end{pmatrix}.$$

Thus we get that all  $v_{i,j}$  are either  $-1$  or  $0$  and it is clear that in any given row  $i_0$  only one of the  $v_{i_0,j}$  can be  $-1$  (as they sum up to  $-s_{i_0} \geq -1$ ). Therefore  $\mathbf{v} \in V_{n,s}$ .  $\square$

**Lemma 17.** *If  $m_{j_0} = 1$  for some  $j_0$  then  $\mathbf{v} \in P_{n,s}$ .*

*Proof.* Since  $m_{j_0}$  is equal to 1, all other  $m_j$  and  $m$  have to be equal to zero. Thus all  $v_{i,j_0}$  are non-negative and at least one of them is equal to 1. Also, all other  $v_{i,j}$  (i.e. if  $j \neq j_0$ ) are non-positive and so are all  $s_i$ . Just as in the proof of Lemma 16 we can show that all  $v_{i,j}$  are either  $-1$ ,  $0$ , or  $1$ ; we omit the details. Furthermore it is clear that if  $v_{i_0,j_0}$  is equal to 1 for some  $i_0$ , then there cannot be more than one  $-1$  in the  $i_0$ -th row (as  $s_{i_0} \leq 0$ ). By the same reasoning, if  $v_{i_0,j_0}$  is equal to 0, there cannot be any  $-1$  in the  $i_0$ -th row at all. Considering the definition of  $P_{n,s}$  we thus see that  $\mathbf{v} \in P_{n,s}$ .  $\square$

**Lemma 18.** *If  $m_j \neq 1$  for all  $j$  then  $m_j = 0$  for all  $j$ .*

*Proof.* We assume to the contrary that  $m_{j_0} \in (0, 1)$  for some  $j_0$ . Suppose that all other  $m_j$  are equal to zero. Then  $m = 1 - m_{j_0} \in (0, 1)$  and for  $\varepsilon := \min\{m_{j_0}, 1 - m_{j_0}\} > 0$  and  $\delta \in [0, \varepsilon]$  we get that  $g_{n,s}(v_{1,1}, \dots, v_{1,j_0} \pm \delta, \dots, v_{1,s}, \dots, v_{n,1}, \dots, v_{n,j_0} \pm \delta, \dots, v_{n,s}) = 1$ , hence  $\mathbf{v}$  is not a vertex of  $\mathcal{E}_{n,s}$ . As an example consider

$$\mathbf{v} = \begin{pmatrix} 0 & 1/3 & 0 & -2/3 \\ 0 & -1/2 & 0 & -1/4 \\ 0 & 1/6 & -1/6 & -2/3 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1/3 \pm \delta & 0 & -2/3 \\ 0 & -1/2 \pm \delta & 0 & -1/4 \\ 0 & 1/6 \pm \delta & -1/6 & -2/3 \end{pmatrix}.$$

If on the other hand there is another  $m_{j_1}$  that is not equal to zero we get  $g_{n,s}(x_{1,1}, \dots, x_{1,j_0} \pm \delta, \dots, x_{1,j_1} \mp \delta, \dots, x_{1,s}, \dots, x_{n,1}, \dots, x_{n,j_0} \pm \delta, \dots, x_{n,j_1} \mp \delta, \dots, x_{n,s}) = 1$  for  $\varepsilon := \min\{m_{j_0}, m_{j_1}, 1 - m_{j_0}, 1 - m_{j_1}\} > 0$  and  $\delta \in [0, \varepsilon]$ , so again  $\mathbf{v}$  is not a vertex of  $\mathcal{E}_{n,s}$ . An example for this situation is

$$\mathbf{v} = \begin{pmatrix} 0 & 1/3 & 0 & -1/6 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 1/6 & 2/3 & -2/3 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1/3 \pm \delta & 0 \mp \delta & -1/6 \\ 0 & -1/2 \pm \delta & 1/2 \mp \delta & 0 \\ 0 & 1/6 \pm \delta & 2/3 \mp \delta & -2/3 \end{pmatrix}.$$

$\square$

*Proof of Theorem 15.* Lemma 18 implies that if  $\mathbf{v}$  is a vertex of  $\mathcal{E}_{n,s}$ , then we are in the situation of either Lemma 16 or Lemma 17, hence  $\mathbf{v} \in P_{n,s} \cup V_{n,s}$ . Furthermore it is clear that  $\mathbf{0}$  is not a vertex of  $\mathcal{E}_{n,s}$ . Also, it follows from the definition of  $P_{n,s}$  that

$$E_{n,s} := (P_{n,s} \cup V_{n,s}) \setminus \{\mathbf{0}\} = ((V_{n,s} - (U_{n,s} \setminus \{\mathbf{0}\})) \cup V_{n,s}) \setminus \{\mathbf{0}\} = (V_{n,s} - U_{n,s}) \setminus \{\mathbf{0}\}.$$

We are left to show that  $E_{n,s} \subseteq V(\mathcal{E}_{n,s})$ . First we observe that  $g_{n,s}(\mathbf{v}) = 1$  for all  $\mathbf{v} \in E_{n,s}$ . We consider the case that  $n, s \geq 2$  and assume that there is a  $\mathbf{v} \in E_{n,s}$  that is not a vertex of  $\mathcal{E}_{n,s}$ . Since  $\mathbf{v}$  is on the boundary of  $\mathcal{E}_{n,s}$  but not a vertex of  $\mathcal{E}_{n,s}$ , it is contained in the interior of an at least 1-dimensional face  $\mathcal{F}$  of  $\mathcal{E}_{n,s}$ . Let  $\mathbf{w}$  be any vertex of  $\mathcal{F}$ . Then  $\mathbf{w} \in E_{n,s}$  and  $\mathbf{v} \neq \mathbf{w}$ .

Now let  $\mathbf{a} \in E_{2,2}$ ,  $\mathbf{a} \neq \mathbf{b} \in E_{2,2} \cup \{\mathbf{0}\}$ , and consider the convex combination  $\alpha\mathbf{a} + (1-\alpha)\mathbf{b}$ ,  $\alpha \in \mathbb{R}$ . By plugging in all possible values of  $\mathbf{a}$  and  $\mathbf{b}$  one can verify that if  $\alpha > 1$  then  $g_{2,2}(\alpha\mathbf{a} + (1-\alpha)\mathbf{b}) > 1$ .

Let  $\mathbf{v}', \mathbf{w}' \in \mathbb{R}^{2 \times s}$  be submatrices consisting of 2 rows of  $\mathbf{v}$  and  $\mathbf{w}$  respectively, such that  $\mathbf{v}' \neq \mathbf{0}$  and  $\mathbf{v}' \neq \mathbf{w}'$ . By definition of  $E_{n,s}$ ,  $\mathbf{v}'$  and  $\mathbf{w}'$  thus respectively contain submatrices of the form  $\mathbf{a}$  and  $\mathbf{b}$  from above while the remaining entries are padded with zeros. It follows from the definition of  $g_{n,s}$  that  $g_{n,s}(\alpha\mathbf{v} + (1-\alpha)\mathbf{w}) > 1$  if  $\alpha > 1$ , which contradicts the fact that  $\mathbf{v}$  is in the interior of  $\mathcal{F}$ . Hence,  $E_{n,s} \subseteq V(\mathcal{E}_{n,s})$  if  $n, s \geq 2$ . If  $n = 1$  and  $s \geq 2$  one can proceed analogously by considering  $\mathbf{a} \in E_{1,2}$ ,  $\mathbf{a} \neq \mathbf{b} \in E_{1,2} \cup \{\mathbf{0}\}$ ; same goes for  $s = 1$  and  $\mathbf{a} \in E_{1,1}$ ,  $\mathbf{a} \neq \mathbf{b} \in E_{1,1} \cup \{\mathbf{0}\}$ .  $\square$

**Corollary 19.** *The number of vertices of  $\mathcal{E}_{n,s}$  is given by  $(s+1)^{n+1} - s - 1$ .*

*Proof.* Follows directly from Theorem 15 and Lemma 14.  $\square$

## 4 Projections of simplotopes

We will establish a relation between the Everest polytope  $\mathcal{E}_{n,s}$  and a special polytope known as simplotope. This relation will allow the comparison of the volumes of the two polytopes even though they are of different dimension.

**Simplotopes.** For  $s \in \mathbb{N}$ , the  $s$ -simplex  $\Delta_s$  is spanned by the points  $(\mathbf{0}, -\mathbf{e}_s(1), \dots, -\mathbf{e}_s(s))$  in  $\mathbb{R}^s$ , with  $\mathbf{e}_s(i)$  the  $i$ -th standard vector in  $\mathbb{R}^s$ , as before. A *simplotope* is a Cartesian product of the form  $\Delta_{s_1} \times \dots \times \Delta_{s_n}$  with positive integers  $s_1, \dots, s_n$ . Note that in the literature, simplotopes are usually defined in a combinatorially equivalent way using the standard  $s$ -simplex spanned by  $(s+1)$ -unit vectors in  $\mathbb{R}^{s+1}$ . We restrict to the case that all  $s_i$  are equal, and we call

$$\mathcal{S}_{n,s} = \underbrace{\Delta_s \times \dots \times \Delta_s}_{n \text{ times}}$$

the  $(n, s)$ -simplotope for  $n, s \in \mathbb{N}$ .

For instance,  $d$ -hypercubes are  $d$ -fold products of line segments, and therefore  $(n, s)$ -simplotopes with  $n = d$  and  $s = 1$ . It is instructive to visualize a point in  $\mathcal{S}_{n,s}$  as an  $n \times s$ -matrix with real entries in  $[0, 1]$ , where the sums of the entries in each row do not exceed 1. One can readily verify that the set of vertices of the  $(n, s)$ -simplotope is equal to  $V_{n,s}$ , as given in the beginning of Section 3. We show next that simplotopes are examples of polytopes with a non-trivial spine.

**Theorem 20.**  *$U_{n,s}$  is a spine of  $V_{n,s} = V(\mathcal{S}_{n,s})$ .*

*Proof.* Let  $\mathbf{u}_0, \dots, \mathbf{u}_s$  denote the elements of  $U_{n,s}$ . More specifically,  $\mathbf{u}_0 := \mathbf{0}$  and

$$\mathbf{u}_j := \begin{pmatrix} -\mathbf{e}_s(j) \\ \vdots \\ -\mathbf{e}_s(j) \end{pmatrix}$$

for all  $j \in \{1, \dots, s\}$ . Fix an arbitrary point  $\mathbf{p} \in \mathcal{S}_{n,s}$ . As we remarked,  $\mathbf{p}$  can be written as an  $(n \times s)$ -matrix with entries in  $[0, 1]$  where each row sum is at most 1. We extend this matrix to an  $(n \times (s+1))$ -matrix  $\mathbf{p}^*$  by adding a 0-th column, whose entries are chosen such that each row sum is exactly one. In the same way, we can define the set  $V_{n,s}^*$  and  $U_{n,s}^* = \{\mathbf{u}_0^*, \dots, \mathbf{u}_s^*\}$  (this operation can be interpreted as lifting the simplotope into a larger space, where it becomes the Cartesian product of standard simplices). We will show that there exist non-negative real numbers  $\lambda_0, \dots, \lambda_{ns}$  such that  $\sum_{i=0}^{ns} \lambda_i = 1$  and

$$\mathbf{p}^* = \sum_{i=0}^s \lambda_i \mathbf{u}_i^* + \sum_{j=s+1}^{ns} \lambda_j \mathbf{v}_j^*,$$

where the  $\mathbf{v}_j^*$  are elements of  $V_{n,s}^*$ . This proves our claim, because by removing the extra columns, this shows that  $\mathbf{p}$  is contained in an  $(ns)$ -simplex that contains all vertices of  $U_{n,s}$  and is therefore in the  $U_{n,s}$ -span in  $V_{n,s}$ .

We construct the required convex combination next. For  $i = 0, \dots, s$ , define  $\lambda_i$  as the minimal entry of  $\mathbf{p}^*$  in column  $i$  and set

$$M^{(s)} := \mathbf{p}^* - \sum_{i=0}^s \lambda_i \mathbf{u}_i^*.$$

$M^{(s)}$  is a matrix with non-negative entries, has at least  $s+1$  zero entries (one per column), and all row sums are equal. For  $j = s+1, \dots, n$ , we will construct matrices  $M^{(j)}$  with the invariant that all entries are non-negative, at least  $j+1$  entries are zero, and all row sums are equal. We use the following procedure: if all entries of  $M^{(j-1)}$  are zero, choose  $\lambda_j = 0$  and  $\mathbf{v}_j^* \in V_{n,s}^*$  arbitrarily. Otherwise, let  $\lambda_j$  be a smallest non-zero entry of  $M^{(j-1)}$ . In each row of  $M^{(j-1)}$ , there exists some smallest (in its row) non-zero entry (because the row sums are all equal, and the matrix is non-zero), and this entry is  $\geq \lambda_j$  because all entries are non-negative. Picking the index of one such smallest non-zero column entry per row, we obtain a point in  $V_{n,s}^*$  (the unique point which has ones at the given indices) which we denote by  $\mathbf{v}_j^*$ , and we set

$$M^{(j)} := M^{(j-1)} - \lambda_j \mathbf{v}_j^*.$$

Indeed,  $M^{(j)}$  has at least one more zero entry than  $M^{(j-1)}$ , has only non-negative entries, and all its row sums are equal.

At the end of this procedure, the  $(n \times (s+1))$ -matrix  $M^{(ns)}$  has  $ns+1$  zero entries. By the pigeonhole principle, at least one row is completely zero. Since all row sums are equal, it follows that  $M^{(ns)}$  is the zero-matrix. Thus, by expanding our construction,

$$M^{(ns)} = \mathbf{p}^* - \sum_{i=0}^s \lambda_i \mathbf{u}_i^* - \sum_{j=s+1}^{ns} \lambda_j \mathbf{v}_j^*$$

is equal to the zero-matrix which yields a linear combination as required. Moreover, the  $\lambda_i$  sum up to one, because the row sums of  $\mathbf{v}^*$  are equal to one. This proves the claim.  $\square$

We remark that essentially the same approach can be applied to more general simplotopes of the form  $\Delta_{s_1} \times \dots \times \Delta_{s_n}$ , proving that they contain a spine of size  $\min\{s_1, \dots, s_n\} + 1$ . We omit details.

**A linear transformation** We call the matrix

$$\Pi_{n,s} := \begin{pmatrix} & -I_s \\ I_{ns} & \vdots \\ & -I_s \end{pmatrix} = \begin{pmatrix} 1 & & & & 0 & -1 & & 0 \\ & \ddots & & & & & \ddots & \\ & & \ddots & & & 0 & & -1 \\ & & & \ddots & & & \vdots & \\ & & & & \ddots & -1 & & 0 \\ & & & & & & \ddots & \\ 0 & & & & & 1 & 0 & -1 \end{pmatrix} \in \mathbb{R}^{(ns) \times ((n+1)s)},$$

where  $I_d$  is the identity matrix of dimension  $d \in \mathbb{N}$ . The  $(n,s)$ -SE-transformation (“SE” stands for “Simplotope  $\leftrightarrow$  Everest polytope”). We show that the name is justified, as it maps the  $(n+1,s)$ -simplotope onto the  $(n,s)$ -Everest polytope.

**Theorem 21.**  $\Pi_{n,s}(V(\mathcal{S}_{n+1,s}) \setminus U_{n+1,s}) = V(\mathcal{E}_{n,s})$ ,  $U_{n+1,s} \setminus \{\mathbf{0}\}$  is a basis of  $\ker(\Pi_{n,s})$  (in particular,  $\Pi_{n,s}U_{n+1,s} = \{\mathbf{0}\}$ ), and  $\mathbf{0}$  is contained in the interior of  $\mathcal{E}_{n,s}$ . In particular,  $\Pi_{n,s}\mathcal{S}_{n+1,s} = \mathcal{E}_{n,s}$ .

*Proof.* Let

$$\mathbf{v} = \begin{pmatrix} -\mathbf{e}_s(j_1) \\ \vdots \\ -\mathbf{e}_s(j_{n+1}) \end{pmatrix} \in V_{n+1,s} \setminus U_{n+1,s}, \quad \mathbf{u} = \begin{pmatrix} -\mathbf{e}_s(j_0) \\ \vdots \\ -\mathbf{e}_s(j_0) \end{pmatrix} \in U_{n+1,s},$$

where  $j_1, \dots, j_{n+1} \in \{0, \dots, s\}$  not all equal and  $j_0 \in \{0, \dots, s\}$ . First we note that any row of  $\Pi_{n,s}$  is of the form

$$\mathbf{r}_{i,j} := (\underbrace{\mathbf{0}, \dots, \mathbf{0}}_{i-1 \text{ times}}, \mathbf{e}_s(j), \underbrace{\mathbf{0}, \dots, \mathbf{0}}_{n-i \text{ times}}, -\mathbf{e}_s(j)) \in \mathbb{R}^{(n+1)s} \simeq \mathbb{R}^{(n+1) \times s},$$

where  $i \in \{1, \dots, n\}$ ,  $j \in \{1, \dots, s\}$ , and  $\mathbf{0}$  is the  $s$ -dimensional zero (row-)vector. For some  $i$  and  $j$ , we can thus compute

$$\mathbf{r}_{i,j} \mathbf{v} = \mathbf{e}_s(j)(-\mathbf{e}_s(j_i)) - \mathbf{e}_s(j)(-\mathbf{e}_s(j_{n+1})) = \delta_{j_{n+1},j} - \delta_{j_i,j},$$

where  $\delta_{x,y}$  for  $x, y \in \mathbb{R}$  is the Kronecker delta function, i.e.  $\delta_{x,y} \in \{0, 1\}$  and  $\delta_{x,y} = 1$  iff  $x = y$ . But then we get

$$\begin{aligned} \Pi_{n,s} \mathbf{v} &= \begin{pmatrix} \delta_{j_{n+1},1} - \delta_{j_1,1} & \dots & \delta_{j_{n+1},s} - \delta_{j_1,s} \\ \vdots & & \vdots \\ \delta_{j_{n+1},1} - \delta_{j_n,1} & \dots & \delta_{j_{n+1},s} - \delta_{j_n,s} \end{pmatrix} = \begin{pmatrix} \delta_{j_{n+1},1} & \dots & \delta_{j_{n+1},s} \\ \vdots & & \vdots \\ \delta_{j_{n+1},1} & \dots & \delta_{j_{n+1},s} \end{pmatrix} - \begin{pmatrix} \delta_{j_1,1} & \dots & \delta_{j_1,s} \\ \vdots & & \vdots \\ \delta_{j_n,1} & \dots & \delta_{j_n,s} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{e}_s(j_{n+1}) \\ \vdots \\ \mathbf{e}_s(j_{n+1}) \end{pmatrix} - \begin{pmatrix} \mathbf{e}_s(j_1) \\ \vdots \\ \mathbf{e}_s(j_n) \end{pmatrix} = \begin{pmatrix} -\mathbf{e}_s(j_1) \\ \vdots \\ -\mathbf{e}_s(j_n) \end{pmatrix} - \begin{pmatrix} -\mathbf{e}_s(j_{n+1}) \\ \vdots \\ -\mathbf{e}_s(j_{n+1}) \end{pmatrix}. \end{aligned}$$

If  $j_{n+1} \neq 0$  we thus get that  $\Pi_{n,s} \mathbf{v} \in V_{n,s} - (U_{n,s} \setminus \{\mathbf{0}\})$  and  $\Pi_{n,s} \mathbf{v} \neq \mathbf{0}$  (since  $j_1, \dots, j_{n+1}$  are not all equal), hence  $\Pi_{n,s} \mathbf{v} \in P_{n,s} \setminus \{\mathbf{0}\}$  by definition of  $P_{n,s}$ . If, on the other hand,  $j_{n+1} = 0$  then  $\Pi_{n,s} \mathbf{v} \in V_{n,s} \setminus \{\mathbf{0}\}$ . Altogether we can conclude

$$\Pi_{n,s}(V(\mathcal{S}_{n+1,s}) \setminus U_{n+1,s}) = \Pi_{n,s}(V_{n+1,s} \setminus U_{n+1,s}) = (P_{n,s} \cup V_{n,s}) \setminus \{\mathbf{0}\} = V(\mathcal{E}_{n,s}).$$

On the other hand we get

$$\mathbf{r}_{i,j} \mathbf{u} = \mathbf{e}_s(j)(-\mathbf{e}_s(j_0)) - \mathbf{e}_s(j)(-\mathbf{e}_s(j_0)) = 0,$$

hence  $\Pi_{n,s} U_{n+1,s} = \{\mathbf{0}\}$ . Clearly  $U_{n+1,s} \setminus \{\mathbf{0}\}$  is linearly independent and  $\mathbf{0}$  is an interior point of  $\mathcal{E}_{n,s}$ .  $\square$

If  $\Pi_{n,s}$  was a square matrix/endomorphisms, we could relate the volumes of  $\mathcal{S}_{n+1,s}$  and  $\mathcal{E}_{n,s}$  simply by the determinant of  $\Pi_{n,s}$ . Yet since dimensions are lost, literally anything can happen to the volume of the image in general – a rotating broomstick might cast shadows of highly varying areas while its volume stays constant. The main goal of Section 5 will be to overcome this problem at least in the special case when the shadow-casting object has a spine that itself only casts a one-point shadow.

## 5 The volume of the Everest polytope

We will present our first main result: a general formula for the volume of the  $(n, s)$ -Everest polytope. The results of this section will also form the basis of our second main result, the characterization of spinal triangulations. However, we will develop our geometric argument only as far as it is necessary to prove the volume formula of the Everest polytope in this section and postpone the generalization to Section 6.

**Foldings and lifts.** In the following we fix a finite full-dimensional point set  $V \subset \mathbb{R}^d$  in convex position, spanning a polytope  $\mathcal{P}$ . Also, let  $U \subseteq V$  be a set of  $n \in \{2, \dots, d\}$  points and assume without loss of generality that  $\mathbf{0}$  is among these points. Setting  $e := d - n + 1$ , we call a linear map  $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^e$  a *U-folding* for  $V$  if the points in  $U$  span the kernel of  $\Phi$ ,  $\Phi(\mathbf{v}) \neq \Phi(\mathbf{w})$  for all  $\mathbf{v}, \mathbf{w} \in V \setminus U$  with  $\mathbf{v} \neq \mathbf{w}$ , and  $\Phi(V \setminus U) \subseteq \mathbb{R}^e$  is a point set in convex position whose convex hull is  $e$ -dimensional and contains  $\mathbf{0}$  in its interior. In this case  $\Phi(\mathcal{P}) = \text{conv}(\Phi(V)) = \text{conv}(\Phi(V) \setminus \{\mathbf{0}\}) = \text{conv}(\Phi(V \setminus U))$ ,  $\Phi(\mathbf{v}) \neq \mathbf{0}$  for all  $\mathbf{v} \in V \setminus U$  and there is a one-to-one correspondence between the points in  $V \setminus U$  and the points in  $\Phi(V \setminus U)$  given by  $\mathbf{v} \mapsto \Phi(\mathbf{v})$ . An example of a *U-folding* is the SE-transformation from Section 4, where  $U$  is the spine of the simplotope, and the images of the non-spine vertices of the simplotope span the Everest polytope in  $\mathbb{R}^e$  as shown in Theorem 21.

In what follows, we fix  $\Phi$  to be a *U-folding* for  $V$ . For notational convenience, we use the short forms  $\hat{x} := \Phi(x)$  and  $\hat{X} := \Phi(X)$  for the images of points and sets in  $\mathbb{R}^d$ .

**Lemma 22.** *The points of  $U$  do not lie on a common facet of  $\mathcal{P}$ . Equivalently, the kernel of  $\Phi$  intersects the interior of  $\mathcal{P}$ .*

*Proof.* Assume to the contrary that some facet of  $\mathcal{P}$  contains the entire set  $U$ , and let  $\mathbb{H}$  be its support. Since  $\Phi$  is a *U-folding* for  $V$ , the kernel of  $\Phi$  is contained in  $\mathbb{H}$ . It follows that  $\hat{\mathbb{H}}$  is an  $(e - 1)$ -dimensional hyperplane passing through the origin which lies in the interior of  $\hat{\mathcal{P}}$ . Thus  $\hat{\mathbb{H}}$  separates two vertices in  $\hat{V}$ , that is, there are  $\mathbf{v}, \mathbf{w} \in V$  such that  $\hat{\mathbf{v}}$  and  $\hat{\mathbf{w}}$  are on opposite sides of  $\hat{\mathbb{H}}$ . Consequently,  $\mathbf{v}$  and  $\mathbf{w}$  lie on different sides of  $\mathbb{H}$ , contradicting the fact that  $\mathbb{H}$  is a supporting hyperplane for  $\mathcal{P}$ .  $\square$

$\hat{V}$  is a point set in convex position plus the origin as an interior point and there is a one-to-one correspondence between the points in  $V \setminus U$  and the points in  $\Phi(V \setminus U)$ . For an  $e$ -simplex  $\hat{\sigma} \subseteq \mathbb{R}^e$  containing  $\mathbf{0}$  as vertex, there is thus a unique *lifted*  $d$ -simplex  $\sigma \subseteq \mathbb{R}^d$  that is spanned by the pre-image of  $V(\hat{\sigma})$ , i.e.  $\sigma = \text{conv}(\Phi^{-1}(V(\hat{\sigma})))$  (note the slight abuse of notation as we chose “ $\hat{\sigma}$ ” as the name of a simplex before even defining the simplex  $\sigma$ , but the naming respects our convention). Any vertex of  $\hat{\sigma}$  that is not the origin has a unique pre-image, while the pre-image of the origin itself is equal to  $U$  by definition, yielding a total of  $e + n = d + 1$  points in  $V$ , including all points of  $U$ . We can relate the volumes of  $\hat{\sigma}$  and  $\sigma$  as follows. Interpret the *U-folding*  $\Phi$  as an  $(e \times d)$ -matrix with respect to the standard bases of  $\mathbb{R}^d$  and  $\mathbb{R}^e$ . By assumption,  $\Phi$  has an  $(e \times e)$ -submatrix with non-zero determinant. Without loss of generality we may assume that this submatrix consists of the first  $e$  columns of  $\Phi$ , and we let  $\Phi_e$  denote this submatrix. Furthermore let  $M_U$  denote the lower  $((d - e) \times (d - e))$ -submatrix of  $(\mathbf{u}_1, \dots, \mathbf{u}_{d-e})$  where  $\mathbf{u}_1, \dots, \mathbf{u}_{d-e}$  are the non-zero elements of  $U$  in some arbitrary order.

**Lemma 23.**

$$d! |\det(\Phi_e)| \text{vol}(\sigma) = e! |\det(M_U)| \text{vol}(\hat{\sigma}).$$

*Proof.* By construction,  $\sigma$  and  $\hat{\sigma}$  both contain the origin (of their corresponding spaces). We define the matrix  $\Phi^*$  as the  $(d \times d)$ -matrix with the first  $e$  rows equal to  $\Phi$ , and the  $i$ -th row equal to  $i$ -th  $d$ -dimensional unit row vector  $\mathbf{e}_d(i)$  for  $i = e + 1, \dots, d$ .  $\Phi^*$  is a block matrix, and its determinant equals  $\det(\Phi_e) -$  the determinant of its upper-left  $(e \times e)$ -block. It follows that

$$\text{vol}(\Phi^*(\sigma)) = |\det(\Phi_e)| \text{vol}(\sigma).$$

Let  $\sigma$  be spanned by the vertices  $\mathbf{v}_1, \dots, \mathbf{v}_e, \mathbf{u}_1, \dots, \mathbf{u}_{d-e}, \mathbf{0}$ , where  $\{\mathbf{u}_1, \dots, \mathbf{u}_{d-e}, \mathbf{0}\} = U$ . The volume of  $\Phi^*(\sigma)$  equals  $1/d! \det(\mathbf{v}_1^*, \dots, \mathbf{v}_e^*, \mathbf{u}_1^*, \dots, \mathbf{u}_{d-e}^*)$ , where  $\mathbf{v}^* = \Phi^*(\mathbf{v})$  for any vertex  $\mathbf{v}$ . By definition of  $\Phi^*$ , this matrix has the form

$$\begin{pmatrix} A & 0 \\ B & M_U \end{pmatrix},$$

where the upper-right block is zero because all  $\mathbf{u}_i$  are in the kernel of  $\Phi$ . Moreover, for  $i \in \{1, \dots, e\}$ , the  $i$ -th column of  $A$  equals  $\hat{\mathbf{v}}_i$ , so that

$$|\det(A)| = |\det(\hat{\mathbf{v}}_1, \dots, \hat{\mathbf{v}}_e)| = e! \text{vol}(\hat{\sigma}).$$

Putting everything together, we obtain

$$d! |\det(\Phi_e)| \text{vol}(\sigma) = |\det(A)| |\det(M_U)| = e! |\det(M_U)| \text{vol}(\hat{\sigma})$$

as claimed.  $\square$

Since  $\hat{V} \setminus \{\mathbf{0}\} = \Phi(V \setminus U)$  is a point set in convex position and since the origin is an interior point of  $\hat{P} = \text{conv}(\hat{V})$ ,  $\hat{V}$  permits a star triangulation as defined in Section 2, that is, a triangulation in which each  $e$ -simplex contains the origin as a vertex. We define the *lift of a star triangulation*  $\hat{\mathcal{T}}$  as the set of all lifted  $e$ -simplices in  $\hat{\mathcal{T}}$ , together with all faces of the obtained lifts.

**Lemma 24.** *The lift of a star triangulation of  $\hat{V}$  is a  $U$ -spinal simplicial complex in  $\mathbb{R}^d$  whose underlying space is a subset of  $\mathcal{P}$ .*

*Proof.* Fix a star triangulation  $\hat{\mathcal{T}}$  and let  $\mathcal{T}$  denote its lift. For any simplex in  $\mathcal{T}$ , all faces are included by construction. We need to show that for two simplices  $\sigma$  and  $\tau$  in  $\mathcal{T}$ ,  $\sigma \cap \tau$  is a face of both. We can assume without loss of generality that  $\sigma$  and  $\tau$  are maximal, hence  $d$ -simplices. By construction,  $\sigma$  and  $\tau$  are the lifts of  $e$ -simplices  $\hat{\sigma}$  and  $\hat{\tau}$  in  $\hat{\mathcal{T}}$ . We can assume that  $\sigma$  and  $\tau$  intersect, as otherwise there is nothing to show. Then, also  $\hat{\sigma}$  and  $\hat{\tau}$  intersect, and since they belong to a triangulation, their intersection is a common face, spanned by a set of vertices  $\{\mathbf{0}, \hat{\mathbf{v}}_1, \dots, \hat{\mathbf{v}}_k\}$ . Let  $\mathbf{v}_i$  be the (unique) pre-image of  $\hat{\mathbf{v}}_i$  under  $\Phi$  for  $i \in \{1, \dots, k\}$  and  $U = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ . Then, the intersection of  $\sigma$  and  $\tau$  is the simplex spanned by the vertices  $\{\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{v}_1, \dots, \mathbf{v}_k\}$  as one can readily verify. This shows that the lift is a simplicial complex. Its underlying space lies in  $\mathcal{P}$  because every lifted simplex does. It is  $U$ -spinal because the lift of every simplex contains  $U$  by definition.  $\square$

**Lemma 25.** *Let  $\mathcal{T}$  denote the lift of a star triangulation of  $\hat{V}$ , and let  $\Phi_e$  and  $M_U$  denote the same matrices as in Lemma 23. Then,*

$$d! |\det(\Phi_e)| \text{vol}\left(\bigcup \mathcal{T}\right) = e! |\det(M_U)| \text{vol}(\hat{\mathcal{P}}),$$

where  $\bigcup \mathcal{T}$  is the underlying space of  $\mathcal{T}$ . In particular, the underlying spaces of the lifts of all star triangulations of  $\hat{V}$  have the same volume.

*Proof.* The statement follows directly from Lemma 23 because the relation holds for any simplex in the star triangulation and its lift.  $\square$

**Lifting theorem and volume formula.** Let  $\mathcal{T}$  be the lift of a star triangulation of  $\hat{V}$ . Is  $\mathcal{T}$  a triangulation of  $V$ ? If  $U$  is not a spine of  $V$ , the answer is no, because all  $d$ -simplices in  $\mathcal{T}$  contain  $U$  by definition, and they cannot cover all of  $\mathcal{P}$  by assumption. The first step towards our main geometric result is that the answer is yes, however, if  $U$  is a spine:

**Theorem 26** (Lifting theorem). *Let  $V \subseteq \mathbb{R}^d$  be a finite point set in convex position and  $U \subseteq V$ . If  $U$  is a spine of  $V$  and  $\Phi$  is a  $U$ -folding for  $V$ , there exists a star triangulation of  $\hat{V}$  whose lift forms a triangulation of  $V$ . In particular,  $V$  has a  $U$ -spinal triangulation.*

Before we expose the details of the rather complicated proof of this theorem, we discuss its consequences. Most importantly, using Lemma 25, we can conclude that

$$d! |\det(\Phi_e)| \text{vol}(\mathcal{P}) = e! |\det(M_U)| \text{vol}(\hat{\mathcal{P}}). \quad (*)$$

We are now ready to prove our main theorem of this section:

**Main Theorem 1.** *Let  $n, s \in \mathbb{N}$ . Then the volume of the  $(n, s)$ -Everest polytope is given by*

$$\text{vol}(\mathcal{E}_{n,s}) = \frac{((n+1)s)!}{(ns)!(s!)^{n+1}}.$$

*Proof.* Let  $U_{n,s}$  denote the spine of the  $(n, s)$ -simplex  $\mathcal{S}_{n,s}$  as defined in Theorem 20. The  $(n, s)$ -SE-transformation  $\Pi_{n,s} : \mathbb{R}^{(n+1)s} \rightarrow \mathbb{R}^{ns}$  is a  $U_{n+1,s}$ -folding for  $\mathcal{S}_{n+1,s}$ , and  $\Pi_{n,s}(\mathcal{S}_{n+1,s}) = \mathcal{E}_{n,s}$  by Theorem 21. Therefore,  $(*)$  relates the volumes of  $\mathcal{E}_{n,s}$  and  $\mathcal{S}_{n+1,s}$ . For  $\Pi_{n,s}$ , we can easily verify that its left  $(ns \times ns)$ -submatrix has determinant 1 and  $M_{U_{n+1,s}}$ , the lower  $(s \times s)$ -submatrix of  $(\mathbf{u}_1, \dots, \mathbf{u}_s)$  where  $\mathbf{u}_1, \dots, \mathbf{u}_s$  are the non-zero elements of  $U_{n+1,s}$ , is a permutation matrix whose determinant is also 1 in absolute value.



Moreover, recall that  $\mathcal{S}_{n+1,s}$  is the  $n+1$ -fold product of simplices  $\Delta_s$  spanned by the origin and  $s$  unit vectors. Hence, the volume of  $\Delta_s$  is  $1/s!$ , and by Fubini's theorem,

$$\text{vol}(\mathcal{S}_{n+1,s}) = (\text{vol}(\Delta_s))^{n+1} = \frac{1}{(s!)^{n+1}}.$$

Together with (\*) this implies

$$((n+1)s)! \frac{1}{(s!)^{n+1}} = (ns)! \text{vol}(\mathcal{E}_{n,s}),$$

which yields the claimed result.  $\square$

**The proof of the lifting theorem.** As before,  $\mathcal{P}$  denotes the polytope spanned by  $V$  and  $n := |U|$ .

**Lemma 27.** *For  $V$ ,  $U$ , and  $\Phi$  as in Theorem 26,  $U$  is a strong spine of  $V$ .*

*Proof.* No facet can contain all points of  $U$  due to Lemma 22.  $\square$

We show next that our point set  $V$  can be transformed through a perturbation into a closeby point set that is in simplicial position such that the perturbed spine remains a spine, and the perturbed folding remains a folding. We already solved the major problem in Theorem 10, that is, to show that simplicial position can be reached without losing the spine property.

**Lemma 28.** *For  $V$ ,  $U$ , and  $\Phi$  as in Theorem 26 and  $\delta > 0$  sufficiently small, there exists a  $\delta$ -perturbation  $\tilde{V}$  of  $V$  such that  $\tilde{U}$  (the set of perturbed points in  $U$ ) is a spine of  $\tilde{V}$ ,  $\Phi$  is a  $\tilde{U}$ -folding for  $\tilde{V}$ , and both  $\tilde{V}$  and  $\Phi(\tilde{V} \setminus \tilde{U})$  are in convex and simplicial position.*

*Proof.* The  $\delta$ -perturbation described in Theorem 10 already has most of the claimed properties so we use the same construction and also use the same notation as in the proof of the theorem. The only thing we need to take care of is to make sure that  $\Phi$  remains a  $\tilde{U}_i$ -folding for  $\tilde{V}_i$  for all  $i \in \{1, \dots, m\}$ . Clearly, for  $\delta_i$  sufficiently small, the origin remains in the interior of the polytope spanned by  $\Phi(\tilde{V}_i \setminus \tilde{U}_i)$ , the image of  $\tilde{V}_i \setminus \tilde{U}_i$  under  $\Phi$  still does not contain the origin and distinct points in  $\tilde{V}_i \setminus \tilde{U}_i$  still have distinct images under  $\Phi$ , so we are left to show that we can guarantee that  $\tilde{U}_i$  still spans the kernel of  $\Phi$ . Since  $\Phi$  is a  $U$ -folding for  $V$ ,  $U$  spans the kernel of  $\Phi$ , and because  $n \geq 2$ , the kernel is at least 1-dimensional. Moreover, there is a line  $\mathbb{L}_1$  contained in the kernel of  $\Phi$  that contains  $\mathbf{u}_1$  and enters the interior of  $\mathcal{P}$  because of Lemma 22. Instead of pulling  $\mathbf{u}_1$  in an arbitrary fashion as in Theorem 10, we pull it along the line  $\mathbb{L}_1$ , i.e. in a way that the perturbed point  $\tilde{\mathbf{u}}_1$  remains on  $\mathbb{L}_1$ . It is clear that the conditions of Theorem 10 are still met (the perturbation  $\tilde{V}_1$  of  $V$  is still a pull of  $\mathbf{u}_1$ ) and that  $\tilde{U}_1$  still spans the kernel of  $\Phi$  ( $\tilde{\mathbf{u}}_1$  stays in  $\mathbb{L}_1$  and thus in the kernel of  $\Phi$ ). By pulling all of the  $\mathbf{u}_i$  in this fashion, it follows by induction that  $\tilde{U}$  spans the kernel of  $\Phi$  (the sufficiently small pushes of the  $\mathbf{v}_i$  have no influence on the kernel of  $\Phi$ ).  $\square$

The next lemma shows that if the lift of a star triangulation fails to cover all of  $\mathcal{P}$ , it will miss a point on the boundary of  $\mathcal{P}$ . In other words, the lift cannot only create “interior holes”. We prove a slightly stronger version for the case that  $\mathcal{P}$  is in simplicial position, as this will be the situation we encounter in the proof of the lifting theorem.

**Lemma 29.** *Let  $V$ ,  $U$  and  $\Phi$  be as in Theorem 26. If  $V$  is in simplicial position and the lift of a star triangulation of  $\hat{V}$  does not cover  $\mathcal{P}$ , there exists a facet of  $\mathcal{P}$  whose interior is not covered.*

*Proof.* It is enough to prove that there exists an uncovered point on the boundary of  $\mathcal{P}$ . By the assumption that the point set is in simplicial position, this ensures that the entire interior of a facet containing the point is not covered.

To see that there exists an uncovered boundary point, fix a simplex  $\hat{\sigma}$  of the star triangulation. Let  $C(\hat{\sigma})$  denote the intersection of the cylindrical set  $\Phi^{-1}(\hat{\sigma})$  with  $\mathcal{P}$ .  $C(\hat{\sigma})$  is convex as it is the intersection of two convex sets. Moreover, the intersection of the lift of the star triangulation and  $C(\hat{\sigma})$  is precisely equal to the lift  $\sigma$  of  $\hat{\sigma}$ , because any other simplex is contained in a different cylindrical set. Since the lift does not cover all of  $\mathcal{P}$ , there exists at least one  $\hat{\sigma}$  such that  $C(\hat{\sigma}) \neq \sigma$ . Pick a point  $\mathbf{p} \in C(\hat{\sigma}) \setminus \sigma$  and choose a line  $\mathbb{L}$  through  $\mathbf{p}$  whose direction vector lies in the kernel of  $\Phi$ .  $\mathbb{L}$  is completely contained in  $\Phi^{-1}(\hat{\sigma})$  and therefore enters and leaves  $C(\hat{\sigma})$  through points  $\mathbf{r}$  and  $\mathbf{s}$  that belong to the boundary of  $\mathcal{P}$ . At least one of the two points  $\mathbf{r}$  and  $\mathbf{s}$  cannot lie in  $\sigma$  because  $\sigma$  is convex and  $\mathbf{p} \notin \sigma$ .  $\square$

*Proof of Theorem 26.* Using Lemma 28, we can assume that  $V$  and  $\Phi(V \setminus U)$  are in simplicial position: if not, we perturb  $V$  by a sufficiently small  $\delta$  such that the perturbed point set  $\tilde{V}$  still satisfies the conditions of the theorem. The (unique) star triangulation of  $\Phi(\tilde{V})$  then induces some star triangulation of  $\hat{V}$  whose lift is a  $U$ -spinal triangulation of  $V$ .

It suffices to show that the lift of the star triangulation covers the entire polytope  $\mathcal{P}$  spanned by  $V$ . Assume to the contrary that this is not the case. Then Lemma 29 implies the existence of a facet  $\mathcal{F}$  of  $\mathcal{P}$  whose interior is missed by the lift. This facet contains exactly  $n - 1$  points of  $U$  by Lemma 27, and  $e$  points  $\mathbf{v}_1, \dots, \mathbf{v}_e \in V \setminus U$ . The bulk of the argument is to show that there exists a point  $\mathbf{v}_0 \in V \setminus U$  not on  $\mathcal{F}$  such that  $\mathbf{v}_0, \dots, \mathbf{v}_e$  span a face of  $\mathcal{P}$ . This face is then contained in some (simplicial) facet of  $\mathcal{P}$  which cannot contain more than  $n - 2$  points of  $U$ , contradicting that  $U$  is a spine of  $V$ .

To certify the existence of  $\mathbf{v}_0$ , let  $\hat{\sigma} \subseteq \mathbb{R}^e$  be the  $e$ -simplex given by the image of  $\mathcal{F}$  under  $\Phi$ .  $\hat{\sigma}$  contains the origin, but does not belong to the star triangulation of  $\hat{V}$  because  $\mathcal{F}$  is not covered by the lift of the star triangulation. It follows that the facet  $\hat{\eta}$  of  $\hat{\sigma}$  that is opposite of the origin is not a facet of  $\hat{\mathcal{P}}$  and must therefore go through its interior (here, we exploit that  $\hat{V}$  is in simplicial position). The support  $\hat{\mathbb{H}}_{\hat{\eta}} \subseteq \mathbb{R}^e$  of  $\hat{\eta}$  strictly separates a non-empty subset  $S$  of  $\hat{V}$  from the origin. We argue next that one element of  $S$  is the projection of the required point  $\mathbf{v}_0$ .

Let  $\mathbb{H}_{\mathcal{F}} \subseteq \mathbb{R}^d$  denote the support of  $\mathcal{F}$ . We let  $\mathbb{H}_{\mathcal{F}}^-$  denote the open half-space that contains all points in  $V \setminus \mathcal{F}$ . The pre-image  $\mathbb{H}_{\hat{\eta}}$  of  $\hat{\mathbb{H}}_{\hat{\eta}}$  under  $\Phi$  is another hyperplane that is not supporting for  $\mathcal{P}$ . The intersection  $\mathbb{A} := \mathbb{H}_{\hat{\eta}} \cap \mathbb{H}_{\mathcal{F}}$  yields a  $(d - 2)$ -dimensional affine subspace of  $\mathbb{H}_{\mathcal{F}}$ , containing the points  $\mathbf{v}_1, \dots, \mathbf{v}_e$  by construction. Moreover, all points in  $U$  are on one side of  $\mathbb{H}_{\hat{\eta}}$ , because  $\Phi$  maps them to  $\mathbf{0}$ , which is not on  $\hat{\mathbb{H}}_{\hat{\eta}}$ . Let  $\mathbb{H}_{\hat{\eta}}^-$  denote the open half-space containing  $U$ , and  $\mathbb{H}_{\hat{\eta}}^+$  the other open halfspace. By construction, the projection of  $\mathbb{H}_{\hat{\eta}}^+ \cap V$  equals  $S$ , so  $\mathbb{H}_{\hat{\eta}}^+ \cap V$  is not empty and contains no points of  $U$ .

We rotate  $\mathbb{H}_{\mathcal{F}}$  along the axis  $\mathbb{A}$  in the direction for which the rotated hyperplane initially remains supporting for  $\mathcal{P}$  (Lemma 11). If we rotate all the way to  $\mathbb{H}_{\hat{\eta}}$ ,  $\mathbb{H}_{\mathcal{F}}$  is transformed to  $\mathbb{H}_{\hat{\eta}}^-$ , because  $U$  remains in  $\mathbb{H}^-$  for all rotated hyperplanes  $\mathbb{H}$  in-between. The rotation direction induces an angular order of the points not on  $\mathbb{H}_{\mathcal{F}}$ . Any point in  $\mathbb{H}_{\hat{\eta}}^+ \cap V$  preceeds any point of  $U$  in that ordering by Lemma 13. Let  $\mathbf{v}_0$  be a minimal point in  $V$  with respect to that ordering. Then  $\mathbf{v}_0 \in \mathbb{H}_{\hat{\eta}}^+ \cap V$  and thus,  $\mathbf{v}_0 \in V \setminus U$ . Let  $\mathbb{H}_0$  denote the hyperplane obtained by rotating  $\mathbb{H}_{\mathcal{F}}$  until we hit  $\mathbf{v}_0$ . By minimality of  $\mathbf{v}_0$ ,  $\mathbb{H}_0$  is still supporting for  $\mathcal{P}$  and contains the points  $\mathbf{v}_0, \dots, \mathbf{v}_e$ . Thus, there is a face of  $\mathcal{P}$  that contains  $\mathbf{v}_0, \dots, \mathbf{v}_e$  by Lemma 12, and consequently, also a facet containing these points. Because  $\mathcal{P}$  is in simplicial position, this facet contains at most  $d - e - 1 = n - 2$  points of  $U$ , contradicting the fact that  $U$  is a spine of  $V$ .  $\square$

## 6 A general lifting process

We will extend the geometric results from Section 5 to obtain a general result on constrained triangulations of polytopes. The first step is a perhaps surprising corollary of the lifting theorem. Recall from Lemma 25 that the underlying spaces of the lifts of all star triangulations of  $\hat{V}$  have the same volume. Since the lifting theorem asserts that at least one such lift triangulates  $\mathcal{P}$ , it follows that all of them triangulate  $\mathcal{P}$ :

**Corollary 30** (Lifting theorem, stronger version). *Let  $V \subseteq \mathbb{R}^d$  be a finite full-dimensional point set in convex position and  $U \subseteq V$ . If  $U$  is a spine of  $V$  and  $\Phi$  is a  $U$ -folding for  $V$ , the lift of any star triangulation of  $\hat{V}$  forms a  $U$ -spinal triangulation of  $V$ .*

This result is useful to algorithmically construct a  $U$ -spinal triangulation of  $\mathcal{P}$  in cases where  $\hat{V}$  is not in simplicial position and its star triangulation not unique: while our proof of the lifting theorem requires a peculiar choice for the star triangulation, caused by a controlled perturbation scheme (as exposed in Lemma 28), the existence of one good triangulation asserts that we can star-triangulate  $\hat{V}$  in an arbitrary fashion and always obtain a  $U$ -spinal triangulation in the lift.

**A special  $U$ -folding.** As before, let  $\mathcal{P}$  be a polytope spanned by a finite full-dimensional point set  $V \subseteq \mathbb{R}^d$  in convex position,  $U \subseteq V$  a spine of  $V$  with  $n = |U| \in \{1, \dots, d + 1\}$  (i.e. every facet of  $\mathcal{P}$  contains at least  $n - 1$  points of  $U$ ), and set  $e := d - n + 1$ . Assume without loss of generality that the origin is among the points in  $U$ . Furthermore let  $\mathbb{A}_U$  be the subspace of  $\mathbb{R}^d$  spanned by  $U$ ,  $\mathbb{A}_U^\perp$  its orthogonal complement,

and  $\Phi_U : \mathbb{R}^d \rightarrow \mathbb{A}_U^\perp$  the (orthogonal) projection of  $\mathbb{R}^d$  to  $\mathbb{A}_U^\perp$ . Since  $U$  is a spine of  $V$ , the dimension of  $\mathbb{A}_U$  is  $n - 1$  and the dimension of  $\mathbb{A}_U^\perp$  is  $e$ . Hence, we may assume without loss of generality that  $\mathbb{A}_U = \mathbb{R}^{n-1}$ ,  $\mathbb{A}_U^\perp = \mathbb{R}^e$ , and  $\Phi_U$  just drops the last  $d - e$  coordinates of a vector. As in Section 5, we make use of the short form  $\hat{x}$  for images under  $\Phi_U$ .

The following lemma might be considered a generalization of Lemma 27.

**Lemma 31.** *The origin is an interior point of  $\hat{\mathcal{P}}$  if and only if  $U$  is a strong spine of  $\mathcal{P}$ .*

*Proof.* For “ $\Rightarrow$ ” we assume to the contrary that there is a facet of  $\mathcal{P}$  that contains all points of  $U \subseteq \ker(\Phi_U)$ . Let  $\mathbb{H}$  be its support. Then  $\hat{\mathbb{H}}$  is a supporting hyperplane for  $\hat{\mathcal{P}}$  and contains the origin. It follows that the origin is on the boundary of  $\mathcal{P}$  which is a contradiction.

For “ $\Leftarrow$ ” we again assume to the contrary that the origin is not in the interior of  $\hat{\mathcal{P}}$ . Then it lies on the boundary and there is a facet of  $\hat{\mathcal{P}}$  that contains the origin. Let  $\hat{\mathbb{H}}$  be its support and  $\mathbb{H} := \Phi_U^{-1}(\hat{\mathbb{H}})$ . Then  $\mathbb{H}$  is a supporting hyperplane for  $\mathcal{P}$  and contains  $U \subseteq \ker(\Phi_U)$ . Hence, there is a facet of  $\mathcal{P}$  which contains all points in  $U$  which is a contradiction.  $\square$

**Lemma 32.** *If  $\mathbf{v} \in V \setminus U$  then  $\hat{\mathbf{v}} \neq \mathbf{0}$ . Furthermore, if  $\mathbf{v} \neq \mathbf{w} \in V \setminus U$  then  $\hat{\mathbf{v}} \neq \hat{\mathbf{w}}$ .*

*Proof.* Since  $U$  is a spine of  $V$ , there is a  $d$ -simplex  $\sigma$  in the  $U$ -span in  $V$  which has  $\mathbf{v}$  among its vertices. If  $\mathbf{v}$  is in the kernel of  $\Phi_U$ , then  $\mathbf{v}$  and the points in  $U$  (which span the kernel of  $\Phi_U$ ) are linearly dependent and  $\sigma$  cannot be full-dimensional, which is a contradiction. Thus  $\hat{\mathbf{v}} \neq \mathbf{0}$ .

Analogously, there is also a  $d$ -simplex  $\sigma$  in the  $U$ -span in  $V$  which has both  $\mathbf{v}$  and  $\mathbf{w}$  among its vertices. If  $\hat{\mathbf{v}} = \hat{\mathbf{w}}$  then the line through  $\mathbf{v}$  and  $\mathbf{w}$  is parallel to the kernel of  $\Phi_U$  and  $\mathbf{v}$ ,  $\mathbf{w}$ , and the points in  $U$  are again linearly dependent, which is again a contradiction. Thus  $\hat{\mathbf{v}} \neq \hat{\mathbf{w}}$ .  $\square$

**Lemma 33.** *If  $n \in \{2, \dots, d\}$  and  $U$  is a strong spine of  $V$ , then  $\Phi_U$  is a  $U$ -folding for  $V$ .*

*Proof.* It is clear that  $U$  spans the kernel of  $\Phi_U$  and that  $\hat{\mathcal{P}}$  is full-dimensional. By Lemma 31 we get that the origin is contained in the interior of  $\hat{\mathcal{P}}$ . In particular, it is not a vertex of  $\hat{\mathcal{P}}$  which implies that it is contained in  $\text{conv}(\Phi_U(V \setminus U))$ . Thus  $\hat{\mathcal{P}} = \text{conv}(\Phi_U(V \setminus U))$  and the origin lies in the interior of  $\text{conv}(\Phi_U(V \setminus U))$ . Lemma 32 implies that the image of  $V \setminus U$  under  $\Phi_U$  does not contain the origin and that distinct points in  $V \setminus U$  have distinct images under  $\Phi_U$ .

It remains to show that  $\Phi_U(V \setminus U)$  is in convex position, and we will do so by contradiction. So, let  $\mathbf{v} \in V \setminus U$  be such that its projection  $\hat{\mathbf{v}}$  lies in  $\text{conv}(\hat{V} \setminus \{\hat{\mathbf{v}}\})$ . Observe that in this case,  $\text{conv}(\hat{V} \setminus \{\hat{\mathbf{v}}\}) = \hat{\mathcal{P}}$ . We argue first that without loss of generality, we can assume that  $V$  is in simplicial position and  $\hat{\mathbf{v}}$  lies in the interior of  $\hat{\mathcal{P}}$ : Recall from Theorem 10 and Lemma 28 that first pulling the points in  $U$  and subsequently pushing the points in  $V \setminus U$  yields simplicial position of  $V$  without affecting the strong spine property or the kernel of  $\Phi_U$ . If  $\hat{\mathbf{v}}$  lies in the interior of  $\hat{\mathcal{P}}$ , we can ensure that the pull and push perturbations are so small that  $\hat{\mathbf{v}}$  remains in the interior. If  $\hat{\mathbf{v}}$  lies on the boundary of  $\hat{\mathcal{P}}$ , we note that the pulls of  $U$  do not affect  $\hat{\mathcal{P}}$  (as the origin remains in the interior), and we push  $\mathbf{v}$  first among all points in  $V \setminus U$ . We claim that after that push, the projection of the perturbed point lies in the interior of  $\hat{\mathcal{P}}$ . To see that, let  $\hat{\mathcal{F}}$  be any such facet of  $\hat{\mathcal{P}}$  that contains  $\hat{\mathbf{v}}$ . Let  $\hat{\mathbb{H}}$  be its support, and set  $\mathbb{H} := \Phi_U^{-1}(\hat{\mathbb{H}})$ . Then  $\mathbb{H}$  is a supporting hyperplane for  $\mathcal{P}$ . Because we push  $\mathbf{v}$ , the perturbed version  $\tilde{\mathbf{v}}$  lies in the interior of  $\mathcal{P}$ , and therefore,  $\tilde{\mathbf{v}}$  also lies in the open halfspace defined by  $\mathbb{H}$  that contains  $\mathcal{P}$ . Consequently, the projection lies in the open halfspace of  $\hat{\mathbb{H}}$  that contains  $\hat{\mathcal{P}}$ , which implies the claim. All further pushes of the remaining points in  $V \setminus U$  are chosen such that the projection of  $\tilde{\mathbf{v}}$  remains an interior point.

Fix  $\mathbf{v} \in V \setminus U$  such that  $\hat{\mathbf{v}}$  is an interior point of  $\hat{\mathcal{P}}$ . To show that this implies a contradiction, let  $\mathcal{F}$  be a facet of  $\mathcal{P}$  that contains  $\mathbf{v}$ . This facet contains exactly  $n - 1$  points of  $U$  and  $e$  points  $\mathbf{v}_1, \dots, \mathbf{v}_e \in V \setminus U$  as  $U$  is a strong spine of  $V$  and  $V$  is in simplicial position. Just as in the proof of Theorem 26, we will show that there exists a point  $\mathbf{v}_0 \in V \setminus U$  not on  $\mathcal{F}$  such that  $\mathbf{v}_0, \dots, \mathbf{v}_e$  span a face of  $\mathcal{P}$ . This face is then contained in some (simplicial) facet of  $\mathcal{P}$  which cannot contain more than  $n - 2$  points of  $U$ , contradicting that  $U$  is a spine of  $V$ .

To find  $\mathbf{v}_0$ , let  $\hat{\sigma} \subseteq \mathbb{R}^e$  be the  $e$ -simplex given by the image of  $\mathcal{F}$  under  $\Phi_U$ .  $\hat{\sigma}$  contains the origin and since  $\hat{\mathbf{v}}$  is an interior point of  $\hat{\mathcal{P}}$ , the facet  $\hat{\eta}$  of  $\hat{\sigma}$  that is opposite of the origin goes through the interior of  $\hat{\mathcal{P}}$ . The support  $\hat{\mathbb{H}}_{\hat{\eta}} \subseteq \mathbb{R}^e$  of  $\hat{\eta}$  strictly separates a non-empty subset  $S$  of  $\hat{V}$  from the origin. Thus we are in the same situation as in the proof of Theorem 26, and by rotating the support  $\mathbb{H}_{\mathcal{F}}$  of  $\mathcal{F}$  around the

axis induced by the pre-image of  $\hat{\mathbb{H}}_{\hat{\eta}}$  under  $\Phi_U$ , we can conclude analogously that one element of  $S$  is the projection of the required point  $\mathbf{v}_0$ .  $\square$

**Theorem 34.** *If  $n \in \{2, \dots, d\}$  and  $U$  is a spine of  $V$ , then there is a  $U$ -spinal triangulation of  $V$ .*

*Proof.* If  $U$  is a strong spine of  $V$ , then  $\Phi_U$  is a  $U$ -folding for  $V$  by Lemma 33. Hence, a  $U$ -spinal triangulation of  $V$  exists by Theorem 26.

If  $U$  is not a strong spine of  $V$ , then we get by Lemma 31 that the origin lies on the boundary of  $\hat{\mathcal{P}}$  and we cannot apply Lemma 33 and Theorem 26 directly. We will overcome this problem by introducing a single auxiliary vertex whose projection will make the origin an interior point. For that, let  $F_U$  be the set of all facets of  $\mathcal{P}$  that contain  $U$  and  $F_{-U}$  the other facets of  $\mathcal{P}$ . Furthermore let  $\mathbf{p}$  be any interior point of the simplex spanned by  $U$ . Then the facets of  $\mathcal{P}$  that contain  $\mathbf{p}$  are exactly the facets in  $F_U$ . We now “pull”  $\mathbf{p}$  from  $\mathcal{P}$  in the sense of Section 2: for any facet  $\mathcal{F}$  of  $\mathcal{P}$  let  $\mathbb{H}_{\mathcal{F}}$  denote the support of  $\mathcal{F}$ ,  $\mathbb{H}_{\mathcal{F}}^+$  the open half-space defined by  $\mathbb{H}_{\mathcal{F}}$  which contains the interior of  $\mathcal{P}$ ,  $\mathbb{H}_{\mathcal{F}}^-$  the complementary open half-space, and let  $\mathbf{v}$  be any point in  $(\bigcap_{\mathcal{F} \in F_U} \mathbb{H}_{\mathcal{F}}^+) \cap (\bigcap_{\mathcal{F} \in F_{-U}} \mathbb{H}_{\mathcal{F}}^-)$ , i.e.  $\mathbf{v}$  is a “pulled version” of  $\mathbf{p}$  from  $\hat{\mathcal{P}}$  but only pulled as far as to not cross any supports of any facets that  $\mathbf{p}$  is not contained in. Then the facets of  $\mathcal{P}$  that are visible from  $\mathbf{v}$  are exactly the facets in  $F_U$ . We define the new point set  $V' := V \cup \{\mathbf{v}\}$  and denote by  $\mathcal{P}'$  its convex hull. The facets of  $\mathcal{P}'$  are the facets in  $F_{-U}$  (which do not change as they are invisible from the only new point  $\mathbf{v}$ ) and additional facets involving  $\mathbf{v}$ . Each such additional facet is obtained by joining  $\mathbf{v}$  with a ridge  $\mathcal{R}$  of  $\mathcal{P}$  such that one of its incident facets is in  $F_U$  and the other is in  $F_{-U}$  [20, p. 113].  $\mathcal{R}$  must contain exactly  $n - 1$  points of  $U$ , so that every facet containing  $\mathbf{v}$  contains exactly  $n - 1$  points of  $U$ , and the same is true for the facets in  $F_{-U}$ . It follows that  $\mathcal{P}'$  is full-dimensional,  $V'$  is in convex position,  $U \subseteq V'$ , and  $U$  is a strong spine of  $V'$ . Hence,  $\Phi_U$  is a  $U$ -folding of  $V'$  by Lemma 33.

Since  $\Phi_U$  is a  $U$ -folding of  $V'$ ,  $\hat{V}'$  is in convex position with the origin in its interior. It follows that either  $\hat{V}$  is also in convex position, or the origin is on the boundary of  $\text{conv}(\hat{V} \setminus \{\mathbf{0}\})$  (it cannot be in the interior because then,  $V$  would be a strong spine). Nevertheless, in both cases, there is a star triangulation  $\hat{\mathcal{T}}$  of  $\hat{V}$ . Since we can choose  $\mathbf{v}$  arbitrarily close to  $\mathbf{p}$ ,  $\hat{\mathbf{v}}$  can be chosen arbitrarily close to  $\hat{\mathbf{p}} = \mathbf{0}$ . Thus, we can assume without loss of generality that the facets of  $\hat{\mathcal{P}}$  that are visible from  $\hat{\mathbf{v}}$  are precisely the facets that contain  $\mathbf{0}$  as vertex. We extend the triangulation  $\hat{\mathcal{T}}$  by adding  $e$ -simplices joining  $\mathbf{v}$  with every visible facet of  $\hat{\mathcal{P}}$ . This yields a star triangulation  $\hat{\mathcal{T}}'$  of  $\hat{V}'$ .

Using the extended star triangulation  $\hat{\mathcal{T}}'$  as auxiliary structure, it is now straight-forward to prove that the lift of  $\hat{\mathcal{T}}$  is a  $U$ -spinal triangulation of  $\mathcal{P}$ . The lift is a  $U$ -spinal simplicial complex by Lemma 24, so it suffices to show that the lift covers  $\mathcal{P}$ . Assume for a contradiction that there exists a point  $\mathbf{q} \in \mathcal{P}$  that is not covered; we can clearly assume that  $\mathbf{q}$  is an interior point of  $\mathcal{P}$ . Since  $\mathcal{P} \subset \mathcal{P}'$ ,  $\mathbf{q}$  is covered by the lift of some simplex of  $\hat{\mathcal{T}}'$  by Corollary 30. Since  $\hat{\mathcal{T}}'$  is an extension of  $\hat{\mathcal{T}}$ , it follows that this simplex has to contain the extra vertex  $\mathbf{v}$ . However, this is a contradiction, because all such simplices are interior-disjoint from  $\hat{\mathcal{P}}$  by construction, so their lift cannot cover  $\mathbf{q}$ .  $\square$

Putting everything together, we obtain our second main result.

**Main Theorem 2** (General lifting theorem). *Let  $\mathcal{P}$  be a polytope spanned by a finite full-dimensional point set  $V \subseteq \mathbb{R}^d$  in convex position and  $U := \{u_1, \dots, u_n\} \subseteq V$ . Then,*

*$U$  is a spine of  $V$  if and only if there exists a  $U$ -spinal triangulation of  $V$ .*

*In this case, for  $n \geq 1$ , the  $U$ -spinal triangulations of  $V$  are exactly the lifts of the star triangulations of  $\hat{V}$ , the orthogonal projection of  $V$  to the orthogonal complement of the affine space spanned by  $U$ . Furthermore, if  $n \geq 2$ ,  $e := d - n + 1$ , and  $A$  is the  $d \times (n - 1)$  matrix with columns  $\mathbf{u}_2 - \mathbf{u}_1, \dots, \mathbf{u}_n - \mathbf{u}_1$ , then*

$$d! \text{vol}(\mathcal{P}) = e! \sqrt{\det(A^T A)} \text{vol}(\hat{\mathcal{P}}).$$

*Proof.* To prove the equivalence, note that both statements are obviously true for  $n \leq 1$  and obviously false for  $n > d + 1$ . For  $n = d + 1$ , both statements are equivalent to  $\mathcal{P}$  being a  $d$ -simplex. Hence, we may assume that  $n \in \{2, \dots, d\}$ . Then, “ $\Rightarrow$ ” is immediately clear from the geometric characterization of spines (Lemma 1), and “ $\Leftarrow$ ” is the statement of Theorem 34. If  $\mathcal{T}$  is a  $U$ -spinal triangulation of  $V$ , then

$\hat{\mathcal{T}} = \{\hat{\sigma} \mid \sigma \in \mathcal{T}\}$  is a star triangulation of  $\hat{V}$  by Lemma 23 (if any two distinct elements of  $\hat{\mathcal{T}}$  did overlap, the volume of  $\bigcup \hat{\mathcal{T}} = \hat{\mathcal{P}}$  would be less than the volume of  $\hat{\mathcal{P}}$ ) and  $\mathcal{T}$  is the lift of  $\hat{\mathcal{T}}$ .

To see the relation between the volumes of  $\mathcal{P}$  and  $\hat{\mathcal{P}}$  we assume without loss of generality that  $\mathbf{u}_1 = \mathbf{0}$ . The matrix  $A^T A$  is known as the *Gram matrix* of the vectors  $u_2, \dots, u_n$ , whose  $ij$ -entry corresponds to the scalar product of  $u_i$  and  $u_j$ , and whose determinant is the squared  $(n-1)$ -dimensional volume of the parallelotope spanned by  $u_2, \dots, u_d$ . Let  $\mathbf{b}_1, \dots, \mathbf{b}_e \in \mathbb{R}^d$  such that  $(\mathbf{b}_1, \dots, \mathbf{b}_e)$  forms an orthonormal basis of the orthogonal complement of the span of  $U$ . Then,  $(\mathbf{b}_1, \dots, \mathbf{b}_e, \mathbf{u}_2, \dots, \mathbf{u}_n)$  is a basis of  $\mathbb{R}^d$  and we let  $B \in \mathbb{R}^{d \times d}$  denote the matrix that corresponds to the change from this basis to the standard basis  $(\mathbf{e}_d(1), \dots, \mathbf{e}_d(d))$  of  $\mathbb{R}^d$ .  $B$  is a regular matrix which maps the parallelotope spanned by  $\mathbf{b}_1, \dots, \mathbf{b}_e, \mathbf{u}_2, \dots, \mathbf{u}_n$  to the unit cube in  $\mathbb{R}^d$ . The absolute value of its determinant is thus given by the reciprocal of the volume of the parallelotope which is an “ $e$ -fold prismatic polytope” with the base being spanned by  $\mathbf{u}_2, \dots, \mathbf{u}_n$  and all  $e$  heights equal to 1 (as  $\mathbf{b}_1, \dots, \mathbf{b}_e$  is orthonormal and orthogonal to any vector of  $U$ ). It follows that  $\text{vol}(\mathcal{P}) = \sqrt{\det(A^T A)} \text{vol}(B\mathcal{P})$ . By construction,  $\text{vol}(\hat{\mathcal{P}}) = \text{vol}(\Phi B\mathcal{P})$  where  $\Phi = (I_e, 0) \in \mathbb{R}^{e \times d}$  is the matrix which projects  $\mathbb{R}^d$  to  $\mathbb{R}^e$  by simply dropping the last  $d-e$  entries. Lemma 23 implies that  $d! |\det(\Phi_e)| \text{vol}(B\mathcal{P}) = e! |\det(M_{BU})| \text{vol}(\Phi B\mathcal{P})$ , where the matrices  $\Phi_e$  and  $M_{BU}$  are defined as described before Lemma 23. In the present situation, we have  $\Phi_e = I_e$  and  $M_{BU}$  is a permutation matrix which simplifies the relation to  $d! \text{vol}(B\mathcal{P}) = e! \text{vol}(\Phi B\mathcal{P}) = e! \text{vol}(\hat{\mathcal{P}})$ , and the claim follows.  $\square$

## 7 Conclusions and further remarks

Main Theorem 2 combines several new results. It settles the question for the existence of a triangulation of a polytope under the constraint that a given subset of the vertices of the polytope must be contained in every maximal simplex of the triangulation. Furthermore, it characterizes all such triangulations and provides a method to compute one (or all) efficiently through the lift of a star triangulation. Finally, it generalizes the well-known relation  $\text{vol}(AM) = |\det(A)| \text{vol}(M)$  where  $M$  is a measurable subset of  $\mathbb{R}^d$  and  $A \in \mathbb{R}^{d \times d}$  to certain cases where  $A$  is not a square matrix. In particular, it allows us to express the volume of an object in  $\mathbb{R}^d$  in terms of the volume its “shadow” in  $\mathbb{R}^e$ , and vice versa.

The shadow that a cube in  $\mathbb{R}^3$  casts if the light shines parallel to any of its space diagonals is a regular hexagon. Assuming a cube with side length  $\ell$ , the theorem implies that the volume of the cube and the volume (area) of its shadow (the hexagon) differ by a factor of  $\sqrt{3}/\ell$  which provides an alternative method to compute the volume of a hexagon from the volume of a cube. By lifting the “complicated” hexagon to a higher dimensional space it gains more symmetries and becomes the comparatively simple cube. In the same fashion, the complicated Everest polytope is the shadow of the simpler simplotope which allowed the computation of its volume in Main Theorem 1.

Starting with a polytope and a spine (with at least two points), it is easy to determine the volume of the “shadow” with respect to the spine using our theorem. On the other hand, there is no easy way to tell if a given shape is the shadow of some higher dimensional object and in the case of the Everest polytope, this is the interesting direction. We pose the question of whether other polytopes (e.g., the Birkhoff polytope) can be expressed as shadows of other polytopes. For that purpose, it might be worthwhile to find general methods or at least good heuristics to determine if a complicated shape can be recognized as the shadow of some simpler object.

Our studies also provide a new type of triangulation of the simplotope. For simplicity, we focus on the case that  $\mathcal{S} := \Delta^1 \times \Delta^2$ , where  $\Delta^1$  and  $\Delta^2$  are  $s$ -simplices for a fixed  $s$ . We obtain a vertex of the simplotope by forming a pair, choosing one vertex from  $\Delta^1$  and one from  $\Delta^2$ . Similarly, we obtain a  $k$ -simplex by picking  $(k+1)$  pairwise distinct pairs in this way. As described in [7], a convenient way of representing such a  $k$ -simplex is as a bipartite graph, where the vertex set is  $V(\Delta^1) \sqcup V(\Delta^2)$ . In our figures, we represent these graphs by drawing the vertices  $\mathbf{0}, -\mathbf{e}_s(1), \dots, -\mathbf{e}_s(s)$  of  $\Delta^1$  from top to bottom on the left and in the same order for  $\Delta^2$  on the right. A triangulation of  $\mathcal{S}$  can then be represented by a collection of bipartite graphs, each with  $2s+1$  edges (because the dimension of  $\mathcal{S}$  is  $2s$ ), with each graph in the collection corresponding to one maximal simplex in the triangulation. It is clear that two maximal simplices have a common facet if and only if their graphs have  $2s$  edges in common.

Recall that  $\{\mathbf{0}, (-\mathbf{e}_s(1), -\mathbf{e}_s(1)), \dots, (-\mathbf{e}_s(s), -\mathbf{e}_s(s))\}$  is a spine of  $\mathcal{S}$ . The spine points are represented by the  $s+1$  horizontal edges in the bipartite graph representation. A spinal triangulation is thus a trian-

gulation such that the representation of each maximal simplex contains the  $s + 1$  horizontal edges. In [7], such triangulations are given for  $s = 2$  (Figure 5) and for  $s = 3$  (Figure 6).

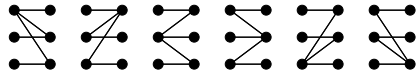


Figure 5: A spinal triangulation for  $\Delta_2 \times \Delta_2$  (taken from [7, p. 305]).

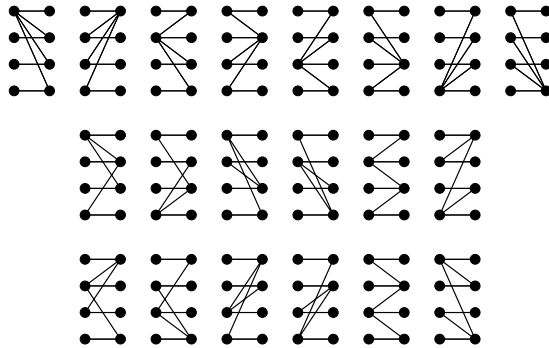


Figure 6: A spinal triangulation for  $\Delta_3 \times \Delta_3$  (taken from [7, p. 306]).

Our results imply that similar constructions exist for all values of  $d$ : just star-triangulate the  $(1, s)$ -Everest polytope with respect to the origin and lift the triangulation to the simplotope. Indeed, we are able to obtain a spinal triangulation of the simplotope for  $s = 10$  in 74.6 seconds on a standard laptop (with a non-optimized implementation), yielding a triangulation with  $\binom{20}{10} = 184756$  maximal simplices. However, the question remains whether there exists a simple combinatorial description of this class of triangulations with respect to their bipartite graph representation. Indeed, such descriptions exist for low dimensions: for  $s = 2$ , the triangulation is simply the set of bipartite trees containing the three horizontal edges. For  $s = 3$ , the triangulation has the property of being closed under swapping left and right and applying even permutations [7]. While we have not found an equally simple description for  $s = 4$ , we found a (rather involved) combinatorial construction of a spinal triangulation for  $s = 4$ ; we give a summary in Figure 7 and omit further details. The study of this class of triangulations for higher  $s$  and  $n$  is an interesting subject of further research.

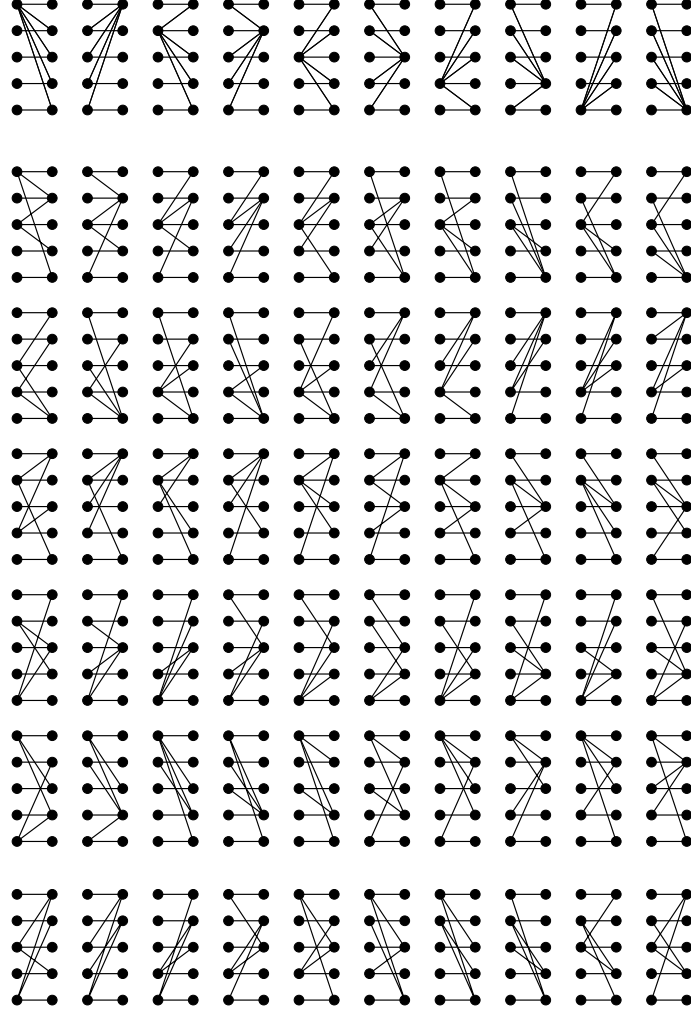


Figure 7: A spinal triangulation for  $\Delta_4 \times \Delta_4$ . The triangulation consists of 70 8-simplices which fall in 3 classes. The first row consists of 10 simplices whose graphs are “star-shapes” (ignoring horizontal edges). The next 50 simplices (rows 2-6) form a “long cycle”, in which two consecutive simplices are adjacent. Note that the topological structure of the graphs in columns 1 and 6 is a simple path (ignoring horizontal edges), whereas this is not true for the graphs in the other columns. The last 10 simplices (row 7) form a short cycle of simple paths. Note also that for the long and short cycle, the second half is the symmetric copy of the first half.

## References

- [1] A. Baker and G. Wüstholz. Logarithmic forms and group varieties. *J. Reine Angew. Math.*, 442:19–62, 1993.
- [2] F. Barroero, C. Frei, and R. Tichy. Additive unit representations in rings over global fields – a survey. *Publ. Math. Debrecen*, 79(3–4):291–307, 2011.
- [3] T. Burger, P. Gritzmann, and V. Klee. Polytope projection and projection polytopes. *Amer. Math. Monthly*, 103(9):742–755, 1996.
- [4] H. Croft, K. Falconer, and R. Guy. *Unsolved Problems in Geometry*. Springer, 1991.

- [5] M. de Berg, O. Cheong, M. v. Kreveld, and M. Overmars. *Computational Geometry – Algorithms and Applications*. Springer, 3rd edition edition, 2008.
- [6] J. de Loera, F. Liu, and R. Yosida. A generating function for all semi-magic squares and the volume of the Birkhoff polytope. *J. Algebraic Combin.*, pages 113–139, 2009.
- [7] J. de Loera, J. Rambau, and F. Santos. *Triangulations*. Springer, 2010.
- [8] H. Edelsbrunner and M. Kerber. Dual complexes of cubical subdivisions of  $\mathbb{R}^n$ . *Discrete Comput. Geom.*, 47(2):393–414, 2012.
- [9] G. R. Everest. A “Hardy-Littlewood” approach to the  $S$ -unit equation. *Compos. Math.*, 70(2):101–118, 1989.
- [10] G. R. Everest. Counting the values taken by sums of  $S$ -units. *J. Number Theory*, 35(3):269–286, 1990.
- [11] J. Evertse and H. P. Schlickewei. A quantitative version of the absolute subspace theorem. *J. Reine Angew. Math.*, 548:21–127, 2002.
- [12] C. Frei, R. Tichy, and V. Ziegler. On sums of  $S$ -integers of bounded norm. *Monatsh. Math.*, 175(2):241–247, 2014.
- [13] H. Freudenthal. Simplicialzerlegung beschränkter Flachheit. *Ann. of Math.*, pages 580–582, 1942.
- [14] R. Freund. Combinatorial theorems on the simplotope that generalize results on the simplex and cube. *Math. Oper. Res.*, 11(1):169–179, 1986.
- [15] R. Gardner. *Geometric Tomography*. Cambridge University Press, 2nd edition edition, 2006.
- [16] I. Gelfand, M. Kapranov, and A. Zelevinsky. *Discriminants, Resultants and Multidimensional Determinants*. Birkhäuser, 2008.
- [17] K. Györy and K. Yu. Bounds for the solutions of  $S$ -unit equations and decomposable form equations. *Acta Arith.*, 123(1):9–41, 2006.
- [18] H. Hadwiger. *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*. Springer, 1957.
- [19] R. Hughes and M. Anderson. Simplicity of the cube. *Discrete Math.*, 158(1–3):99–150, 1996.
- [20] P. McMullen and G. Shephard. *Convex polytopes and the upper bound conjecture*. London Mathematical Society, 1971.
- [21] J. Matoušek. *Lectures in Discrete Geometry*. Springer, 2002.
- [22] J. Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, 1999.
- [23] K. Nishioka. Algebraic independence by Mahler’s method and  $S$ -unit equations. *Compositio Math.*, 92(1):87–110, 1994.
- [24] I. Pak. Four questions on Birkhoff polytope. *Ann. Comb.*, 4:83–90, 2000.
- [25] H. P. Schlickewei.  $S$ -unit equations over number fields. *Invent. Math.*, 102(1):95–108, 1990.
- [26] R. Schneider. *Convex bodies: The Brunn-Minkowski theory*. Cambridge University Press, 2nd edition edition, 2014.
- [27] J. Shewchuk. General-dimensional constrained Delaunay and constrained regular triangulations, I: Combinatorial properties. *Discrete Comput. Geom.*, 39:580–637, 2008.
- [28] G. van der Laan and A. Talman. On the computation of fixed points in the product space of unit simplices and an application to noncooperative  $n$  person games. *Math. Oper. Res.*, 7(1):1–13, 1982.
- [29] G. Ziegler. *Lectures on Polytopes*. Springer, 2007.